Fraïssé-like constructions of compacta

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Based on joint work with Wiesław Kubiś,
and on joint work with Tristan Bice and Alessandro Vignati

Abstract Fraïssé theory overview

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Theorem (characterization of the Fraïssé limit)

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and let U be an $\mathcal{L}$-object. Then the following are equivalent.

1. U is cofinal and homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$,
2. U is cofinal and has the extension property in $\langle \mathcal{K}, \mathcal{L} \rangle$,
3. U is the $\mathcal{L}$-limit of a Fraïssé sequence in $\mathcal{K}$.

Moreover, such U is unique and cofinal in $\mathcal{L}$, and every $\mathcal{K}$-sequence with $\mathcal{L}$-limit U is Fraïssé in $\mathcal{K}$.

Theorem (existence of a Fraïssé sequence)

Let $\mathcal{K} \neq \emptyset$ be a category. $\mathcal{K}$ has a Fraïssé sequence if and only if

1. $\mathcal{K}$ is directed,
2. $\mathcal{K}$ has the amalgamation property,
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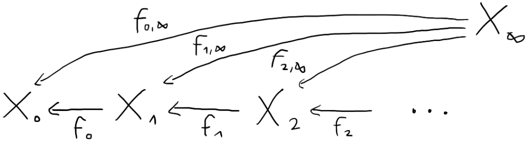

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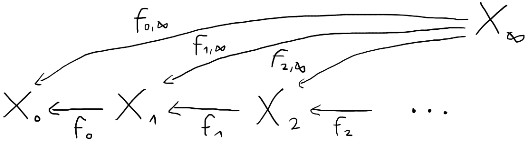

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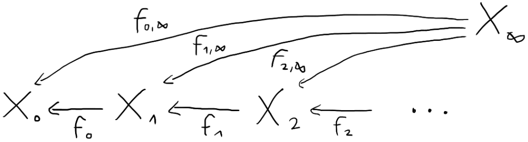

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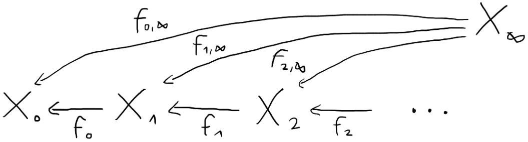

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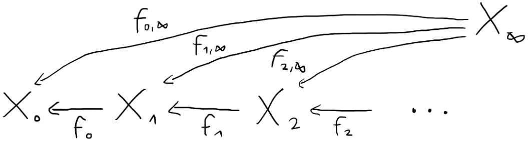

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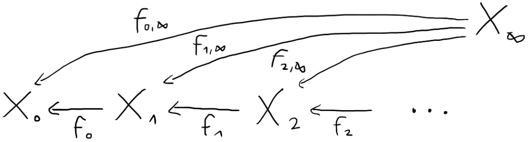

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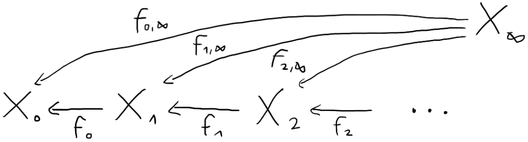

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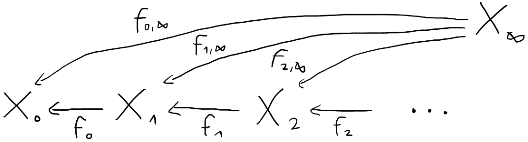

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- $f_{n,\infty} \colon X_n \leftarrow X_\infty$ is the restriction of the projection.

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How to obtain non-zero-dimensional spaces?

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Can we obtain the desired compactum directly as a Fraïssé limit?

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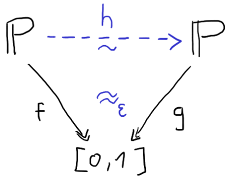

Fraïssé theory of MU-categories overview

Theorem (characterization of the Fraïssé limit)

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free MU-completion and let U be an $\mathcal{L}$-object. Then the following are equivalent.

1. U is cofinal and homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$,
2. U is cofinal and injective in $\langle \mathcal{K}, \mathcal{L} \rangle$,
3. U is the $\mathcal{L}$-limit of a Fraïssé sequence in $\mathcal{K}$.

Moreover, such U is unique and cofinal and homogeneous in $\mathcal{L}$, and every $\mathcal{K}$-sequence with $\mathcal{L}$-colimit U is Fraïssé in $\mathcal{K}$.

Theorem (existence of a Fraïssé sequence)

Let $\mathcal{K} \neq \emptyset$ be an MU-category. $\mathcal{K}$ has a Fraïssé sequence if and only if

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- Let $\mathcal{S}_P$ be the category consisting of the unit circle and all continuous surjections whose degree uses only primes from a set P.

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However, our small objects are not finite any more.

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- Let $\mathcal{K}$ denote the corresponding category.

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- A sequence $\langle X_*, R_* \rangle$ in $\mathcal{K}$ induces an ω-poset $\mathbb{P} = \bigsqcup_{n \in \omega} X_n$ where $\langle n, y \rangle \geq \langle m, x \rangle$ if $n \leq m$ and $y R_{n,m} x$.

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- $\mathbb{P}$ induces the spectrum $\mathrm{S}\mathbb{P} = \{S \subseteq \mathbb{P} : S \text{ minimal selector}\}$, where $S \subseteq \mathbb{P}$ is a selector if S is upwards closed and $S \cap X_n \neq \emptyset$ for every $n \in \omega$.

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- Every second-countable T_1-compactum can be obtained this way.

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But instead of taking the limit, we introduce an ad hoc construction.

Summary

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classical projective FT	✓	✓	✗

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spectra of ω-posets	✓	✗*	✓

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- Let us put $X_\infty = \mathrm{S}\mathbb{P}$ and $yR_{n,\infty}x$ iff $y \in x$.
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- ... this is work in progess.

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