SMALLEST SUBFAMILIES OF MEAGER IDEALS ENSURING P-LIKE PROPERTIES

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Ideal on a set

A family $\mathcal{I} \subseteq \mathcal{P}(X)$ of subsets of a given (usually countable) set X is called an **ideal on $\boldsymbol{X}$**, if

- $[X]^{<\omega} \subseteq \mathcal{I}$, $X \notin \mathcal{I}$,
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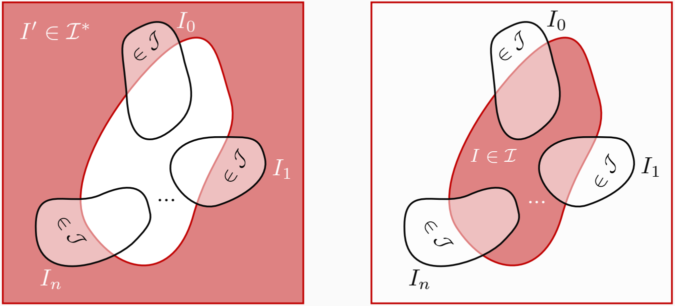
We can identify X with ω via a fixed bijection $\rightarrow$ every time we talk about ideals on X, we are, in fact, talking about **ideals on ω**.

Topology

- $\mathcal{P}(\omega)$ is endowed with the Polish compact topology homeomorphic to the Cantor topology on ${}^{\omega}2$ via characteristic functions
- ${}^{\omega}2$ is being viewed as the product of infinitely many copies of $\{0, 1\}$ (with discrete topology) endowed with Tychonoff product topology
- an ideal $\mathcal{I}$ on ω is meager if it is meager as a subset of the Cantor space $\mathcal{P}(\omega)$

P($\mathcal{J}$)-property

Let $\mathcal{J}$ be an ideal on X. An ideal $\mathcal{I}$ on X is said to be a **P($\mathcal{J}$)-ideal**, if for each countable family $\{I_n : n \in \omega\} \subseteq \mathcal{I}$ there is an $I' \in \mathcal{I}^*$ such that $I_n \cap I' \in \mathcal{J}$ for every $n \in \omega$.



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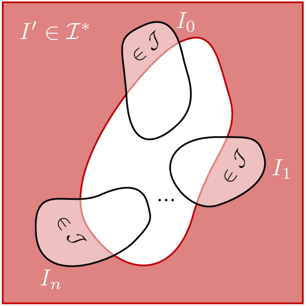

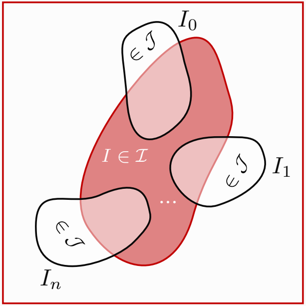

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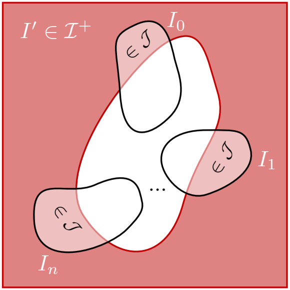
Weak P($\mathcal{J}$)-ideal

Let $\mathcal{J}$ be an ideal on X. An ideal $\mathcal{I}$ on X is said to be a **weak P($\mathcal{J}$)-ideal**, if for each countable family $\{I_n : n \in \omega\} \subseteq \mathcal{I}$ there is an $I' \in \mathcal{I}^+$ such that $I_n \cap I' \in \mathcal{J}$ for every $n \in \omega$.



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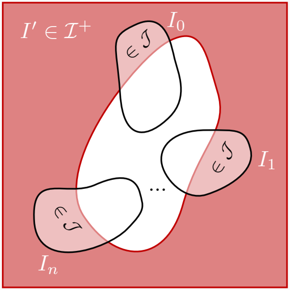
Let $\mathcal{J}$ be an ideal on X. An ideal $\mathcal{I}$ on X is said to be a **weak P($\mathcal{J}$)-ideal**, if for each countable family $\{I_n : n \in \omega\} \subseteq \mathcal{I}$ there is an $I' \in \mathcal{I}^+$ such that $I_n \cap I' \in \mathcal{J}$ for every $n \in \omega$.



Clearly, if $\mathcal{I}$ is a P($\mathcal{J}$)-ideal, then it is a weak P($\mathcal{J}$)-ideal as well.

Encoded partitions

- Sequences of sets can be equivalently replaced by partitions.

We call a function $f\colon \omega \to \omega$ **$\mathcal{I}$-to-one** if $f^{-1}[\{n\}] \in \mathcal{I}$ for any $n \in \omega$.

(M. Repický [1, 2], 2021)

Let $\mathcal{J}$ be an ideal on ω. An ideal $\mathcal{I}$ on ω is said to be a **P($\mathcal{J}$)-ideal**, if for each $\mathcal{I}$-to-one function f there is an $I' \in \mathcal{I}^*$ such that $f^{-1}[\{n\}] \cap I' \in \mathcal{J}$ for every $n \in \omega$.

Similarly for weak P($\mathcal{J}$)-ideals.

Repický M., *Spaces not distinguishing ideal convergences of real-valued functions*, Real Anal. Exch. **46(2)** (2021), 367–394.

Repický M., *Spaces not distinguishing ideal convergences of real-valued functions II*, Real Anal. Exch. **46(2)** (2021), 395–422.

First appearance

Theorem (M. Mačaj – M. Sleziak [1], 2010)

Let X be a non-discrete first countable topological space and let $\mathcal{I},\mathcal{J}$ be ideals on ω. The following are equivalent:

1) *$\mathcal{I}$ is a* $\mathrm{P}(\mathcal{J})$*-ideal.*
2) *In the Boolean algebra $\mathcal{P}(\omega)/\mathcal{J}$ the ideal $\mathcal{I}$ corresponds to a σ-directed subset.*
3) *For any sequence $\langle x_n : n\in\omega\rangle$ in X, if $\langle x_n : n\in\omega\rangle$ is $\mathcal{I}$-convergent to x then $\langle x_n : n\in\omega\rangle$ is $\mathcal{I}^{\mathcal{J}}$-convergent to x.*

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- Later: papers concerning various types of convergence spaces and related cardinal characteristics

Constructing non-P($\mathcal{J}$)-ideals

The basic questions arising:

- For which ideals $\mathcal{J}$ there is a non-P($\mathcal{J}$)-ideal (and how does it look like)?
- Is there some standard way of constructing non-P($\mathcal{J}$)-ideals or at least proving the existence?

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A family $\mathcal{B} \subseteq \mathcal{I}$ is a **base** of $\mathcal{I}$, if $\mathcal{B}$ is a cofinal subset of $\langle \mathcal{I}, \subseteq \rangle$.

$$\mathtt{cof}(\mathcal{I}) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base of } \mathcal{I}\}.$$

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An ideal $\mathcal{I}$ on ω is **tall** if there is no set $Y \in [\omega]^\omega$ such that $\mathcal{I} \restriction Y = [Y]^{<\omega}$, where $\mathcal{I} \restriction Y = \{I \cap Y : I \in \mathcal{I}\}$.

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If there is not a finite subfamily $\mathcal{C}$ of $\mathcal{E}$ with finite $\omega \setminus \bigcup \mathcal{C}$ then an **ideal generated by $\mathcal{E}$** is the smallest ideal containing $\mathcal{E}$ and Fin and we denote this ideal by $\langle \mathcal{E} \rangle$, i. e.

$$\langle \mathcal{E} \rangle = \left\{ E \in \mathcal{P}(M) : E \subseteq^* \bigcup \mathcal{E}' \text{ for some } \mathcal{E}' \in [\mathcal{E}]^{<\omega} \right\}.$$

Constructing non-P($\mathcal{J}$)-ideals

M . Talagrand and S.-A. Jalali-Naini theorem + using behavior of MAD-generated ideals $\rightarrow$

Theorem

*For any meager ideal $\mathcal{J}$ on ω there is a tall meager non-*P($\mathcal{J}$)*-ideal $\mathcal{I}$ with $\mathtt{cof}(\mathcal{I}) = \mathfrak{c}$.*

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- **Intuitive observation:** "size" from 4 points of view:

  topological and P($\mathcal{J}$): negligible/not "dense"

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- Most of the critical ideals appearing in the literature **are meager**.

Introducing cardinal invariant

We wish to describe a size of a smallest family ensuring $\mathrm{P}(\mathcal{J})$-property for any $\mathrm{P}(\mathcal{J})$-ideal $\mathcal{I}$, i.e.,

$$\mathtt{cof}^{\mathcal{J}}_{\mathtt{ct}}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \land (\forall \mathcal{C} \in [\mathcal{I}]^{\omega})(\exists A \in \mathcal{A})(\forall C \in \mathcal{C})\, C \subseteq^{\mathcal{J}} A\}.$$

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- **Observation 1:** If $\mathcal{I}$ is a $\mathrm{P}(\mathcal{J})$-ideal, then to study $\mathtt{cof}^{\mathcal{J}}_{\mathtt{ct}}(\mathcal{I})$ means to study $\mathtt{cof}^{\mathcal{J}}(\mathcal{I})$.

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- **Observation 1:** If $\mathcal{I}$ is a P($\mathcal{J}$)-ideal, then to study $\texttt{cof}^{\mathcal{J}}_{\texttt{ct}}(\mathcal{I})$ means to study $\texttt{cof}^{\mathcal{J}}(\mathcal{I})$.
- **Observation 2:** $\texttt{cof}^{\mathcal{J}}(\mathcal{I}) = \mathfrak{d}(\mathcal{I}, \subseteq^{\mathcal{J}})$, therefore it can be seen as a modification of the $\texttt{cof}^*$-invariant.

Introducing cardinal invariant

Lemma

Let $\mathcal{I}, \mathcal{J}, \mathcal{J}'$ be ideals on ω. Then the following holds true.

a *If $\mathcal{I} \subseteq \mathcal{J}$, then $\mathtt{cof}^{\mathcal{J}}(\mathcal{I}) = 1$, in particular, $\mathtt{cof}^{\mathcal{I}}(\mathcal{I}) = 1$.*

b $\mathtt{cof}^{\mathcal{J}}(\mathcal{I}) \leq \mathtt{cof}(\mathcal{I})$.

c *If $\mathcal{J} \subseteq \mathcal{J}'$, then $\mathtt{cof}^{\mathcal{J}'}(\mathcal{I}) \leq \mathtt{cof}^{\mathcal{J}}(\mathcal{I})$.*

d *Either $\mathtt{cof}^{\mathcal{J}}(\mathcal{I}) = 1$ or $\mathtt{cof}^{\mathcal{J}}(\mathcal{I}) \geq \omega$.*

e *If $\mathcal{I}$ is a $\mathrm{P}(\mathcal{J})$-ideal, then either $\mathtt{cof}^{\mathcal{J}}(\mathcal{I}) = 1$ or $\mathtt{cof}^{\mathcal{J}}(\mathcal{I}) \geq \omega_1$.*

Example

For $\mathcal{J} = \mathrm{Fin}$ we have

Proposition (see. e.g. [1])

$\mathtt{cof}^{\mathrm{Fin}}(\mathcal{I}) = \mathtt{cof}(\mathcal{I})$ *for any uncountably generated ideal* $\mathcal{I}$.

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Consider the following standard construction of a P-ideal:

- Take an increasing chain in $\mathcal{P}(\omega)/\mathrm{Fin}$ of length ω_1 with a chain of representatives $b = \langle B_\alpha : \alpha < \omega_1 \rangle$, s.t. $B_\alpha \subseteq^* B_\beta$ and $|B_\beta \setminus B_\alpha| = \omega$ for any $\alpha < \beta < \omega_1$.

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- $\mathcal{I} := \langle \{B_\alpha : \alpha < \omega_1\} \rangle$ is a P(Fin)-ideal by regularity of ω_1.
- $\mathtt{cof}(\mathcal{I}) = \omega_1 \rightarrow$ using the previous Proposition we have $\mathtt{cof}^{\text{Fin}}(\mathcal{I}) = \omega_1$.

Critical ideals on $\omega \times \omega$

$\mathbf{Fin}$ $= [\omega \times \omega]^{<\omega}$,

$\mathbf{Fin} \times \emptyset$ $= \{I \subseteq \omega \times \omega : (\forall^\infty n < \omega)\ \{m : \langle n, m\rangle \in I\} = \emptyset\}$,
ideal generated by columns

$\emptyset \times \mathbf{Fin}$ $= \{I \subseteq \omega \times \omega : (\forall n < \omega)\ \{m : \langle n, m\rangle \in I\}$ is finite$\}$,
ideal generated by sets of points below graphs of functions from ${}^\omega\omega$

$\mathbf{Fin} \times \mathbf{Fin}$ $= \{I \subseteq \omega \times \omega : (\forall^\infty n < \omega)\ \{m : \langle n, m\rangle \in I\}$ is finite$\}$,
supremum of $\{\mathrm{Fin} \times \emptyset, \emptyset \times \mathrm{Fin}\}$

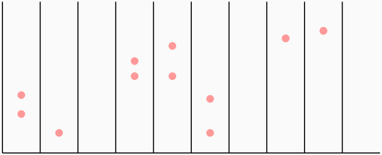
$\boldsymbol{\mathcal{S}el}$ $= \{I \subseteq \omega \times \omega : (\exists k < \omega)(\forall n < \omega)\ |\{m : \langle n, m\rangle \in I\}| < k\}$,
ideal generated by ${}^\omega\omega$

$\boldsymbol{\mathcal{ED}}$ $= \{I \subseteq \omega \times \omega : (\exists k < \omega)(\forall^\infty n < \omega)\ |\{m : \langle n, m\rangle \in I\}| < k\}$.
supremum of $\{\mathrm{Fin} \times \emptyset, \mathcal{S}el\}$

Illustration

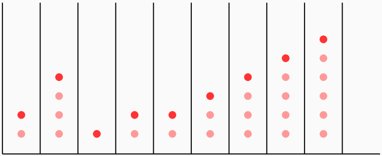
A standard set of $\mathcal{S}el$:

$I \in \mathcal{S}el$



A standard basic set of $\emptyset \times \mathrm{Fin}$:

$I \in \emptyset \times \mathrm{Fin}$



Critical ideals on $\omega \times \omega$

We wish to find a value of $\mathtt{cof}^{\mathcal{J}}(\mathcal{I})$ for every reasonable pair $\mathcal{I}, \mathcal{J}$ of ideals $\mathrm{Fin}, \mathrm{Fin} \times \emptyset, \emptyset \times \mathrm{Fin}, \mathrm{Fin} \times \mathrm{Fin}, \mathcal{S}el, \mathcal{ED}$, that is, for every pair for which we have a positive mark in the following table.

	P	P(Fin × ∅)	P(∅ × Fin)	P(Fin × Fin)	P($\mathcal{S}el$)	P($\mathcal{ED}$)
Fin	✓	✓	✓	✓	✓	✓
Fin × ∅	✗	✓	✗	✓	✗	✓
∅ × Fin	✓	✓	✓	✓	✓	✓
Fin × Fin	✗	✓	✗	✓	✗	✓
$\mathcal{S}el$	✗	✗	✓	✓	✓	✓
$\mathcal{ED}$	✗	✗	✗	✓	✗	✓

Table: $\mathrm{P}(\mathcal{J})$ interactions between critical ideals on $\omega \times \omega$

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	P	P($\mathrm{Fin} \times \emptyset$)	P($\emptyset \times \mathrm{Fin}$)	P($\mathrm{Fin} \times \mathrm{Fin}$)	P($\mathcal{S}el$)	P($\mathcal{ED}$)
Fin	✓	✓	✓	✓	✓	✓
$\mathrm{Fin} \times \emptyset$	✗	✓	✗	✓	✗	✓
$\emptyset \times \mathrm{Fin}$	✓	✓	✓	✓	✓	✓
$\mathrm{Fin} \times \mathrm{Fin}$	✗	✓	✗	✓	✗	✓
$\mathcal{S}el$	✗	✗	✓	✓	✓	✓
$\mathcal{ED}$	✗	✗	✗	✓	✗	✓

Table: P($\mathcal{J}$) interactions between critical ideals on $\omega \times \omega$

$\downarrow \mathcal{I}$ / $\mathcal{J} \rightarrow$	Fin	$\mathrm{Fin} \times \emptyset$	$\emptyset \times \mathrm{Fin}$	$\mathrm{Fin} \times \mathrm{Fin}$	$\mathcal{S}el$	$\mathcal{ED}$
Fin	1	1	1	1	1	1
$\mathrm{Fin} \times \emptyset$		1		1		1
$\emptyset \times \mathrm{Fin}$	$\mathfrak{d}$	$\mathfrak{d}$	1	1	$\mathfrak{d}$	$\mathfrak{d}$
$\mathrm{Fin} \times \mathrm{Fin}$		$\mathfrak{d}$		1		$\mathfrak{d}$
$\mathcal{S}el$			1	1	1	1
$\mathcal{ED}$				1		1

Table: Cardinal $\mathtt{cof}^{\mathcal{J}}_{\mathtt{ct}}(\mathcal{I}) = \mathtt{cof}^{\mathcal{J}}(\mathcal{I})$ for all pairs of ideals considered in this work.

Twofold behavior?

All $\texttt{cof}^{\mathcal{J}}(\mathcal{I})$-numbers for every particular pair of ideals mentioned was either 1 or the cofinality of an ideal itself. Is it true in general?

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Let $\mathcal{I}, \mathcal{J}$ be arbitrary two uncountably generated ideals on ω such that $\omega_1 \leq \texttt{cof}(\mathcal{J}) < \texttt{cof}(\mathcal{I})$ and consider the disjoint sum

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So, it may happen that

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A bit different idea for weak P($\mathcal{J}$)-property

Recall the definition

Let $\mathcal{J}$ be an ideal on X. An ideal $\mathcal{I}$ on X is said to be a **weak P($\mathcal{J}$)-ideal**, if for each countable family $\{I_n : n \in \omega\} \subseteq \mathcal{I}$ there is an $I' \in \mathcal{I}^+$ such that $I_n \cap I' \in \mathcal{J}$ for every $n \in \omega$.

So an ideal $\mathcal{I}$ is **not** a weak P($\mathcal{J}$)-ideal, if there is a countable family $\{I_n : n \in \omega\} \subseteq \mathcal{I}$ such that for any $I' \in \mathcal{I}^+$ there is n s.t. $I_n \cap I' \notin \mathcal{J}$.

Now consider

$$\mathsf{cov}^+(\mathcal{I}) = \min\{|\mathcal{E}| : \mathcal{E} \subseteq \mathcal{I} \ \wedge \ (\forall I' \in \mathcal{I}^+)(\exists I \in \mathcal{E}) \ |I \cap I'| = \omega\}.$$

B. Farkas and L. Zdomskyy, *Ways of destruction*, J. Symb. Log., **87(3)** (2022), 938-966.

O. Guzmán-González., M. Hrušák, C. A. Martínez-Ranero and U. A. Ramos-García, *Generic existence of MAD families,* J. Symb. Log. **82(1)** (2017), 303-316.

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Clearly, an ideal $\mathcal{I}$ is not a weak P(Fin)-ideal iff $\mathsf{cov}^+(\mathcal{I}) \leq \omega$.

Proposition (A.M., M. Repický)

$\mathsf{cov}^+(\langle\mathcal{A}\rangle)$ *is uncountable* ($\geq \mathfrak{b}$) *for any MAD family* $\mathcal{A}$.

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Recall that $\mathcal{I} \leq_K \mathcal{J}$ iff there is $f \in {}^\omega\omega$ s.t. $f^{-1}[I] \in \mathcal{J}$ for all $I \in \mathcal{I}$.

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$\mathrm{Fin} \times \mathrm{Fin} \not\leq_K \langle\mathcal{A}\rangle$ *for any MAD family* $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

Moreover, if $\mathcal{A}$ is a MAD family, then $\langle\mathcal{A}\rangle$ is tall, thus, $\langle\mathcal{A}\rangle \not\leq_K \mathrm{Fin}$.

Consequence

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However, we know that $\mathrm{Fin} \leq_K \langle \mathcal{A} \rangle \leq_K \mathrm{Fin} \times \mathrm{Fin}$, therefore

$$\mathrm{Fin} <_K \langle \mathcal{A} \rangle <_K \mathrm{Fin} \times \mathrm{Fin}.$$

Thank you

Marton A., P-*like properties of meager ideals and cardinal invariants*, manuscript submitted for publication.

Marton A., Šupina J., *On P-like ideals induced by disjoint families*, manuscript submitted for publication, arXiv:2212.07260.