Universally Sacks-indestructible combinatorial families of reals

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joint work with Vera Fischer

03.02.2023

'Pleaser enter an amount divisible by 0'

Introduction

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◦ One aspect of combinatorial set theory is the study of possible sizes of subsets of reals which satisfy some combinatorial property.
◦ Examples: maximal almost disjoint families, maximal independent families, ultrafilter bases on ω, dominating families, ...
◦ We call such types of families 'combinatorial families of reals'.

Example: Mad families

∘ One of the most well-studied type are maximal almost disjoint families:

Definition

A subset $\mathcal{A} \subseteq [\omega]^\omega$ is called almost disjoint (ad) iff for all $A \neq B \in \mathcal{A}$ the set $A \cap B$ is finite and no finite subset of $\mathcal{A}$ almost covers ω, i.e. for all $\mathcal{A}_0 \in [\mathcal{A}]^{<\omega}$ we have that $\omega \setminus \bigcup \mathcal{A}_0$ is infinite.

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Definition

The corresponding spectrum and cardinal characteristic are defined as

$$\text{spec}(\mathit{mad}) := \{|\mathcal{A}| \mid \mathcal{A} \text{ is a mad family}\}$$
$$\mathfrak{a} := \min(\text{spec}(\mathit{mad})).$$

Forcing in combinatorial set theory

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◦ Common forcing applications are:

- Destroying the maximality families (by adding intruders),
- Generating new maximal families (by adding diagonalizing reals),
- Analyse which forcings may preserve the maximality of ground model families.

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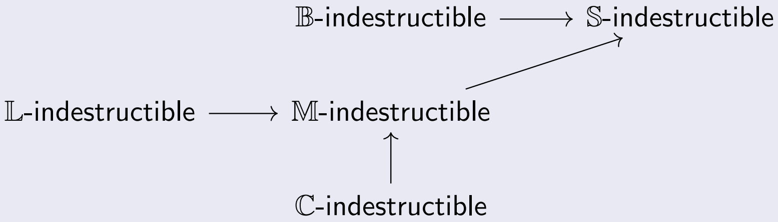

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$$\begin{array}{ccccc} & & \mathbb{B}\text{-indestructible} & \longrightarrow & \mathbb{S}\text{-indestructible} \\ & & & \nearrow & \\ \mathbb{L}\text{-indestructible} & \longrightarrow & \mathbb{M}\text{-indestructible} & & \\ & & \uparrow & & \\ & & \mathbb{C}\text{-indestructible} & & \end{array}$$

Theorem (Brendle, Hrušák, 2002)

Assume $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. *Then there is a* $\mathbb{S}$*-indestructible mad family.*

Universally Sacks-indestructible families

Theorem (Fischer, Schrittesser, 2021)

In L there is a Π^1_1 *maximal eventually different family which is indestructible by any countably supported product or iteration of Sacks-forcing of any length.*

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- We call such combinatorial families of reals universally Sacks-indestructible.

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Under CH *there is a* $\mathbb{S}^{\aleph_0}$*-indestructible maximal eventually different family, where* $\mathbb{S}^{\aleph_0}$ *is the countable support product of Sacks-forcing of size* $\aleph_0$.

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Theorem

Every $\mathbb{S}^{\aleph_0}$*-indestructible maximal eventually different family is universally Sacks-indestructible.*

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◦ In order to generalize these results we have to specify what exactly we mean with a type of combinatorial family:

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◦ In particular we want the property of what constitutes a family of our type and what constitutes an intruder to be arithmetically definable in the following sense:

Arithmetical types

Definition

An arithmetical type $\mathfrak{t}$ (of a combinatorial family of reals) is a pair of sequences $\mathfrak{t} = ((\psi_n)_{n<\omega}, (\chi_n)_{n<\omega})$ such that both $\psi_n(w_0, w_1, \ldots, w_n)$ and $\chi_n(v, w_1, \ldots, w_n)$ are arithmetical formulas in $n+1$ real parameters.

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The domain $\mathrm{dom}(\mathfrak{t})$ of the type $\mathfrak{t}$ is the set

$$\{\mathcal{F} \subseteq \mathcal{P}(\omega^\omega) \mid \forall n < \omega \ \forall \{f_0, \ldots, f_n\} \in [\mathcal{F}]^{n+1} \text{ we have } \psi_n(f_0, \ldots, f_n)\}.$$

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$$\forall n < \omega \forall \{f_1, \ldots, f_n\} \in [\mathcal{F}]^n \ \chi_n(g, f_1, \ldots, f_n)$$

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we call g an intruder for $\mathcal{F}$. If a forcing $\mathbb{P}$ satisfies

$$\mathbb{P} \Vdash \mathcal{F} \text{ has no intruders,}$$

we say $\mathbb{P}$ preserves $\mathcal{F}$ or $\mathcal{F}$ is $\mathbb{P}$-indestructible.

Translating forcing statements in $\mathbb{S}^{\aleph_0}$

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Lemma (Fischer, S., 2022)

Let $\chi(v_1, \ldots, v_k, w_1, \ldots, w_l)$ be an arithmetical formula in $k + l$ real parameters. Further, let $p \in \mathbb{S}^{\aleph_0}$, $f_1, \ldots, f_l$ be reals and $g_1, \ldots, g_k$ be codes for continuous functions $g_i^* : {}^\omega({}^\omega 2) \to {}^\omega\omega$. Then the following are equivalent:

1. $p \Vdash_{\mathbb{S}_{\aleph_0}} \chi(g_1^*(s_{\dot{G}}), \ldots, g_k^*(s_{\dot{G}}), f_1, \ldots, f_l)$,
2. $\forall q \leq p \; \exists r \leq q \; \forall x \in [r] \; \chi(g_1^*(x), \ldots, g_k^*(x), f_1, \ldots, f_l)$.

Universal Sacks-indestructibility

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Proof.

Sketch. Let $\mathbb{P}$ be any product or iteration of Sacks-forcing and assume $g^*(s_{\dot{G}})$ was a name for an intruder for a family $\mathcal{F}$, where $g^* : {}^{\omega}({}^{\omega}2) \to {}^{\omega}\omega$.

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$$\mathbb{S}^{\aleph_0} \Vdash g^*(s_{\dot{G}}) \text{ is not an intruder for } \mathcal{F}$$

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which is by definition of intruder expressed by

$$\mathbb{S}^{\aleph_0} \Vdash \exists n < \omega\ \exists \{f_1, \ldots, f_n\} \in [\mathcal{F}]^n\ \neg\chi_n(g^*(s_{\dot{G}}), f_1, \ldots, f_n).$$

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Since χ_n is an arithmetical formula by the previous Lemma choose $q \leq p$ such that

$$\forall x \in [q]\ \neg\chi_n(g^*(x), f_1, \ldots, f_n).$$

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Now, this is a Π^1_1-formula, so it also holds in the extension by $\mathbb{P}$, which may be used to obtain a contradiction to the assumption that $\mathbb{P}$ forces $g^*(s_{\dot{G}})$ to be an intruder for $\mathcal{F}$. □

Universal Sacks-indestructibility

Corollary (Fischer, S., 2022)

Every $\mathbb{S}^{\aleph_0}$-indestructible mad family/med family/independent family/ultrafilter basis/maximal cofinitary group/partition of Baire space into compact sets/... is universally Sacks-indestructible.

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Corollary (Shelah, Laver, resp.)

Every selective independent family and selective ultrafilter is universally Sacks-indestructible.

Constructing a universally Sacks-indestructible med family

Theorem (Fischer, Schrittesser, 2021)

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Lemma (Fischer, Schrittesser, 2021)

Let $p \in \mathbb{S}^{\aleph_0}$, $\mathcal{F}$ be a countable med family and $\dot{g}$ be a $\mathbb{S}^{\aleph_0}$-name such that

$$p \Vdash \mathcal{F} \cup \{\dot{g}\} \text{ is a med family.}$$

Then there is $q \leq p$ and $f \in {}^{\omega}\omega$ such that $\mathcal{F} \cup \{f\}$ is a med family and

$$q \Vdash \mathcal{F} \cup \{f, \dot{g}\} \text{ is not a med family.}$$

Excluding intruders lemma

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Then there is $q \leq p$ and $f \in {}^{\omega}\omega$ such that $\mathcal{F} \cup \{f\}$ is of type $\mathfrak{t}$ and

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Theorem (Fischer, S., 2022)

Assume CH *and the excluding-intruders-lemma holds for type* $\mathfrak{t}$. *Then there is a universally Sacks-indestructible family of type* $\mathfrak{t}$.

Universally Sacks-indestructible cofinitary group

Lemma (Fischer, S., 2022)

The excluding-intruders-lemma holds for maximal cofinitary groups.

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Corollary (Fischer, S., 2022)

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Corollary (Fischer, S., 2022)

Assume CH. *Then there is a universally Sacks-indestructible maximal cofinitary group.*

We know that universally Sacks-indestructible mad families, med families and partitions of Baire space into compact sets may be constructed in a similar fashion, however:

Question

Does the excluding-intruders-lemma also hold for independent families and ultrafilters?

Thank you for your attention!