Topological universality of the automorphism groups of uncountable Fraïssé limits

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Definition

We will say that $\mathcal{K}$ is a *Fraïssé class* (of length κ), if

- $\mathcal{K}$ has at most κ many objects (up to isomorphism),
- $\mathcal{K}$ has at most κ embeddings,
- $\mathcal{K}$ has the Joint Embedding Property,
- $\mathcal{K}$ has the Amalgamation Property,
- $\mathcal{K}$ is closed under taking co-limits of length $< \kappa$.

Examples:

- The class of all finite structures in a given countable language (groups, rational metric spaces, linear orders etc.)
- The uncountable Fraïssé classes, if $\kappa = \kappa^{<\kappa}$,
- Projective Fraïssé classes,
- Some more exotic classes...

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- The class of all finite structures in a given countable language (groups, rational metric spaces, linear orders etc.)
  - class of finite linear orders $\mapsto (\mathbb{Q}, \leq)$,
  - class of finite graphs $\mapsto$ the random graph,
  - class of finite rational metric spaces $\mapsto$ the rational Urysohn space
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  - class of finite linear orders $\mapsto (\mathbb{Q}, \leq)$,
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  - class of finite rational metric spaces $\mapsto$ the rational Urysohn space
- The uncountable Fraïssé classes, if $\kappa = \kappa^{<\kappa}$,
  - class of linear orders of size $< \kappa \mapsto$ the unique κ saturated linear order of size κ,
  - class of graphs of size $< \kappa \mapsto$ the κ saturated graph of size κ,
  - $\ldots$
- Projective Fraïssé classes,
- Some more exotic classes...

The countable case

The classical case: $\kappa = \omega$, and $\mathcal{K}$ a class of finite models in some relational language, together with all embeddings. We denote by $\mathbb{K}$ the corresponding Fraïssé limit.

The countable case

For "typical" classes $\mathcal{K}$, whenever $A \subseteq \mathbb{K}$, we have an embedding

$$\operatorname{Aut} A \hookrightarrow \operatorname{Aut} \mathbb{K},$$

due to existence of the *Katětov functors*. This is even an embedding of topological groups with the pointwise convergence topology.

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"Typically" means in particular:

- the rational Urysohn space (Uspenskij, 1990),
- the random tournament (Jaligot, 2007),
- any $\mathbb{K}$ for $\mathcal{K}$ with the *Free Amalgamation Property* (Bilge-Melleray, 2013).

The uncountable case

The uncountable case: If $\kappa > \omega$ this is usually not true.

Theorem (Doucha, 2015)

Assume $\omega < \kappa = \kappa^{<\kappa}$. Let M be one of the following structures:

- *κ-saturated graph of size κ,*
- *κ-saturated partial order of size κ,*
- *κ-saturated linear order of size κ,*
- *κ-saturated tournament of size κ,*
- *κ-saturated group of size κ.*

There exists a substructure $A \subseteq M$ such that $\mathrm{Aut}\, A$ does not continuously embed into $\mathrm{Aut}\, M$.

The uncountable case

Theorem (Doucha, 2015)

Assume CH. Then $\operatorname{Aut}\mathbb{Q}$ *does not embed continuously into the automorphism group of the* ω_1*-saturated linear order of size* ω_1.

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But, it turns out there is more to be said.

The uncountable case

The category of linear orders with countable I-dimension.

Definition (Novák, 1964)

A linear order L has a *countable dimension* if

$$L \hookrightarrow [0,1]^{\alpha},$$

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Proposition (K., 2020)

The class of linear orders of countable dimension, together with increasing mappings, forms a Fraïssé class.

Its limit is isomorphic to:

$$\mathbb{L}_{\omega_1} = \{x \in [-1,1]^{\omega_1} \mid \; |\{\alpha < \omega_1 : \; x(\alpha) \neq 0\}| < \omega_1\}.$$

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Theorem (Harzheim)

The order $\mathbb{L}_{\omega_1}$ is a unique ω_1-saturated linear order, which embeds into any ω_1-saturated linear order.

Corollary

If CH holds, then $\mathbb{L}_{\omega_1}$ is the unique ω_1-saturated linear order of size 2^{ω}.

Proposition

If D is a compact line, then

$$D \hookrightarrow \mathbb{L}_{\omega_1}$$

if and only if D has a countable dimension.

Theorem (K.,2020)

If D is a compact line of a countable dimension, then

$$\mathrm{Aut}\, D \hookrightarrow \mathrm{Aut}\, \mathbb{L}_{\omega_1}$$

as a topological group.

Proof.

$$D \times \mathbb{L}_{\omega_1} \simeq \mathbb{L}_{\omega_1}$$

□

The category of linear orders with left-invertible, order preserving mappings

Left-invertible mappings

A function $f : L_1 \hookrightarrow L_2$ is left-invertible if there exists $r : L_2 \to L_1$ such that $r \circ f = \mathrm{id}_{L_1}$.
We will say that L_1 is an *increasing retract* of L_2.

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Theorem (Kubiś, 2014)

K is a Fraïssé class (regardless of CH!).

Left-invertible mappings

Definition

A linear order L is ω_1*-retractible* if

$$L = \bigcup_{\alpha<\omega_1} L_\alpha,$$

where each L_α is a countable increasing retract of L.

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Left-invertible mappings

Theorem (Kubiś, 2014)

There exists a unique up to isomorphism ω_1-retractible linear order $\mathbb{Q}_{\omega_1}$, such that

1. *Each ω_1-retractible linear order is an increasing retract of $\mathbb{Q}_{\omega_1}$,*
2. *Each automorphism between countable increasing retracts of $\mathbb{Q}_{\omega_1}$ extends to an automorphism of $\mathbb{Q}_{\omega_1}$.*

Left-invertible mappings

$$\mathbb{Q}_{\omega_1} = \{x \in \mathbb{Q}^{\omega_1} \mid |\{\alpha < \omega_1 : \ x(\alpha) \neq 0\}| < \omega\}.$$

Proposition

For any countable linear order L there exists a continuous embedding

$$\mathrm{Aut}\, L \hookrightarrow \mathrm{Aut}\, \mathbb{Q}_{\omega_1}.$$

Proof.

$$L \times \mathbb{Q}_{\omega_1} \simeq \mathbb{Q}_{\omega_1}.$$

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Left-invertible mappings

The category of Boolean algebras with regular embeddings

Let us fix a strongly inaccessible cardinal λ, and denote by $\mathcal{B}_\lambda$ the category of all Boolean algebras of size $< \lambda$, together with *regular* embeddings. Let $\mathbb{B}_\lambda$ be the corresponding Fraïssé limit.

Proposition

$\mathcal{B}_\lambda$ is a Fraïssé category.

What can we say about $\mathbb{B}_\lambda$?

Proposition

The Boolean algebra $\mathbb{B}_\lambda$ is the unique Boolean algebra such that:

1. $\mathbb{B}_\lambda = \bigcup_{\alpha<\lambda} A_\alpha$, *where* $A_\alpha \sqsubseteq \mathbb{B}_\lambda$, *and* $|A_\alpha| < \lambda$, *for each* α.
2. *(universality) Each Boolean algebra of size* $< \lambda$ *embeds into* $\mathbb{B}_\lambda$ *as a regular subalgebra.*
3. *(regular injectivity) For all pairs of Boolean algebras* $B \sqsubseteq C$, *where* $|C| < \lambda$, *each regular embedding* $i : B \hookrightarrow \mathbb{B}_\lambda$ *can be extended to a regular embedding* $\bar{i} : C \hookrightarrow \mathbb{B}_\lambda$.

Theorem (Kripke)

If B is a Boolean algebra with a dense subset of size at most κ, then

$$\overline{B \oplus \mathrm{Coll}(\omega, \kappa)} \simeq \mathrm{Coll}(\omega, \kappa).$$

In particular, the algebra $\mathrm{Coll}(\omega, \kappa)$ is universal (in the sense of regular embeddings) for Boolean algebras of size $\leq \kappa$.

Proposition

1. *The algebra* $\mathbb{B}_\lambda$ *can be represented as*

$$\mathbb{B}_\lambda = \bigcup_{\alpha<\lambda} \mathrm{Coll}(\omega, \kappa_\alpha),$$

whenever $(\kappa_\alpha)_{\alpha<\lambda}$ *is an unbounded in* λ *sequence of cardinals.*

2. *The algebra* $\mathbb{B}_\lambda$ *is isomorphic to the free product*

$$\bigoplus_{\delta<\lambda} \mathrm{Coll}(\omega, \delta).$$

Aut $\mathbb{B}_\lambda$ as a topological group.

Proposition

For all pairs of Boolean algebras B, C, the natural embedding

$$B \hookrightarrow B \oplus C$$

induces an embedding of topological groups

$$\mathrm{Aut}\, B \hookrightarrow \mathrm{Aut}\, B \oplus C.$$

$\mathrm{Aut}\,\mathbb{B}_\lambda$ as a topological group.

Proposition

If B is a Boolean algebra that can be decomposed into an increasing chain of regular subalgebras of size $< \lambda$, then

$$\mathrm{Aut}\,B \hookrightarrow \mathrm{Aut}\,\mathbb{B}_\lambda$$

as a topological group.

$\operatorname{Aut}\mathbb{B}_\lambda$ as a topological group.

Proposition

If B is a Boolean algebra that can be decomposed into an increasing chain of regular subalgebras of size $< \lambda$, then

$$\operatorname{Aut} B \hookrightarrow \operatorname{Aut}\mathbb{B}_\lambda$$

as a topological group.

$$\operatorname{Sat} B = \min\{\gamma \in \mathrm{Card} \mid B \text{ does not have an antichain of size } \gamma\}.$$

Corollary

If $\operatorname{Sat} B < \lambda$, then

$$\operatorname{Aut} B \hookrightarrow \operatorname{Aut}\mathbb{B}_\lambda$$

as a topological group.

$\operatorname{Aut}\mathbb{B}_\lambda$ as a topological group.

Thank you for your attention. Ideas and suggestions mostly welcome (crazy ones especially!).