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On some methods of extending measures

Mariam Beriashvili

I. Vekua Institute of Applied Mathematics
Tbilisi State Univeristy
Georgian Technical University

mariam_beriashvili@yahoo.com

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Measure invariant extension problem

One of the first works devoted to (countably additive) invariant extensions of the Lebesgue measure was the paper by Marczewski

- Marczewski E., Sur Vextension de la mesure lebesguienne, Fund. Math., vol. 25, 1935, pp. 551 - 558.

where several constructions of such extensions were considered.
Also, we can point out another paper

- Marczewski E., On problems of the theory of measure, Uspekhi Mat. Nauk, vol. 1, no. 2 (12), 1946, pp. 179 - 188 (in Russian).

by the same author, in which a list of important problems from measure theory was given and, in particular, certain invariant extensions of the Lebesgue measure were touched upon.

Measure invariant extension problem

Among other problems, Marczewski formulates a problem of the existence of a non-separable invariant extension of the measure λ. This problem was solved in 1950 when two papers - by Kakutani and Oxtoby

- Kakutani S., Oxtoby J., Construction of a nonseparable invariant extension of the Lebesgue measure space, Ann. Math., vol. 52, 1950, pp. 580 - 590.

and by Kodaira and Kakutani

- Kodaira K., Kakutani S., A nonseparable translation-invariant extension of the Lebesgue measure space, Ann. Math., vol. 52, 1950, pp. 574 - 579.

- appeared, in which two essentially different constructions of non-separable invariant extensions of λ were presented.

Auxiliary notions

Let E be an uncountable set, $\mathcal{S}$ be a σ-algebra of subsets of E and let μ be a σ-finite measure on E with $\mathrm{dom}(\mu) = \mathcal{S}$. For any two sets $X \in \mathcal{S}$ and $Y \in \mathcal{S}$ satisfying the relations $\mu(X) < +\infty$ and $\mu(Y) < +\infty$, we may put

$$d(X, Y) = \mu(X \triangle Y).$$

The function d is a quasi-metric (pseudo-metric) on S and, after appropriate factorization, yields the metric space canonically associated with μ. The topological weight of this metric space is called the weight of μ.

A measure μ is called non-separable if the above-mentioned metric space is non-separable (i.e., the weight of μ is strictly greater than the first infinite cardinal ω).

Auxiliary notions

Let E be a nonempty set, G be a group of transformations of E and let μ_1 be a σ-finite G-invariant measure defined on some σ-algebra of subsets of E. We recall that the measure μ_1 has the uniqueness property if, for any σ-finite G-invariant measure μ_2 defined on $\mathrm{dom}(\mu_1)$, there exists a coefficient $t \in \mathbf{R}$ (certainly, depending on μ_2) such that $\mu_2 = t \cdot \mu_1$(in other words, μ_1 and μ_2 are proportional measures).

Auxiliary notions

Let again E be a nonempty set and let G be a group of transformations of E. We say that a G-invariant measure μ is metrically transitive with respect to G (or ergodic with respect to G) if, for each μ-measurable set X with $\mu(X) > 0$, there exists a countable family $(g_n)_{n \in N} \subset G$ satisfying the equality

$$\mu(E \setminus \cup_{n \in N} g_n(X)) = 0.$$

Example

For instance, every Haar measure has the uniqueness property in the mentioned sense. It is easy to see that if a σ-finite G-invariant measure μ has the uniqueness property then it is also metrically transitive with respect to the whole group G. The converse assertion is not true in general,only in the case, when μ is a complete σ-finite G-invariant measure.

Metrical transitivity of Haar measure

Lets E be an arbitrary σ-compact locally topological group. Obviously, we may equip E with the σ-finite left Haar measure, which will be denoted by μ. If H is a subset of E, then these two assertions are equivalent:

1. H is dense in E;
2. The measure μ is metrically transitive with respect to H.

Treating the real line R as a vector space over the field Q of all rational numbers and keeping in mind the existence of a Hamel basis of R, it is not difficult to show that the additive group $(R, +)$ admits a representation in the form

$$R = G + H, (R \cap H = \{0\})$$

, where G and H are some subgroups of $(R, +)$ and

$$card(G) = \omega_1, card(H) \leq c$$

.

We denote by I the σ-ideal generated by all those subsets X of R which are representable in the form

$$X = Y + H$$

where $Y \subset G$ and $card(Y) \leq \omega$.
I is a translation-invariant σ-ideal of sets in R.

Lemma 1

Let μ be an arbitrary σ-finite translation-invariant measure on R. There exists a measure μ' on R such that:

1. μ' is translation invariant;
2. μ' extends μ
3. $I \subset dom(\mu')$ and $\mu'(Z) = 0$ for each $Z \in I$

Theorem 1

Let μ be a nonzero σ-fnite translation-invariant measure on R Then the inequality

$$card(M_R(\mu)) \geq 2^{\omega_1}$$

holds true. In particular, there are measures on R strictly extending μ and invariant under the group of all translations of R.

Where $M_R(\mu)$ denotes the family of all measures on R extending μ and invariant with respect to R.

Remark: Let us consider n-dimensional Euclidean space R^n, where $n \geq 1$. Since there exists an isomorphism between the additive groups $(R, +)$ and $(R^n, +)$, the direct analog of above mentioned theorem hold for the space R^n.

Let the symbol λ denote again the standard Lebesgue measure on the real line R. As we have above mentioned, Kakutani and Oxtoby demonstrated in 1950 that there exist nonseparable measures on R belonging to the class $M_R(\lambda)$. Obviously, all those measures are strict (proper) extensions of λ. A radically different approach to the problem of the existence of nonseparable measures belonging to $M_R(\lambda)$ was given in the work of Kodaira and Kakutani. The method of Kakutani and Oxtoby allows one to conclude that there exist at least 2^{2^c} nonseparable measures on R, all of which extendλ and are translation invariant. Thus, for the Lebesgue measure λ on R, the inequality of Theorem 1 can be essentially strengthened and, in fact, we have the following relation:

$$card(M_R(\mu)) = 2^{2^c}$$

In the paper

- M. Beriashvili, A. Kirtadze, "On the uniquiness property of Non-separable extensions of invariant measures and relative Measurabili of real valured functions" Georgian Mathematical Journal, Vol. 21, Issue 1, 2014, pp. 49-57

was demonstrate the next result:

Theorem 2

The cardinal number of the class of all invariant, non-separable measures on the space R^N, which extend the Borel measure χ and posses the uniqueness property, is equal to 2^{2^c}

Surjective homomorphism

Let (G_1, μ_1) and (G_2, μ_2) be any two groups endowed with σ-finite invariant measures and let

$$\phi : G_1 \to G_2$$

be a surjective homomorphism. Suppose that a general property $P(X)$ of a set $X \subset G_2$ is given. Sometimes, it turns out, that

$$P(\phi^{-1}(X)) \iff P(X).$$

In such a situation we say that $P(X)$ is stable under surjective homomorphism.

In particular, if ϕ coincides with the canonical surjective homomorphism

$$Pr_2 : H \times G_2 \to G_2$$

than we may apply the method of direct products, where $H \subset G_1$ and the role of G_1 is play by $H \times G_2$.

Example

Let G be an arbitrary group and let $Y \subset G$. We say that, Y is G - absolutely negligible set in G if, for any σ-finite G-invariant measure μ on G, there exists G-invariant measure μ' on G extending μ and $\mu'(Y) = 0$.

By using the method of surjective homomorphisms, it was shown that for any uncountable commutative group $(G, +)$ there exists two G - absolutely negligible subsets A and B such that their algebraic sum $A + B$ coincides the whole of G.

In the paper

- M. Beriashvili, A. Kirtaze, "Absolutely negligible sets and their algebraic sums", Transactions of A. Razmadze Mathematical Institute Vol. 177 (2023)

we have shown, that the next statement is true:

Theorem 3

Let $(G, \cdot)$ an $(H, \cdot)$ be arbitrary uncountable groups and let

$$\varphi : G \to H$$

be a surjective homomorphism. Let μ be a nonzero σ-finite H-left invariant measure on H. If there exist a nonzero σ-finite H-left invariant (H-left-quasi-invariant) measure $\mu' \supset \mu$ on H and two absolutely negligible sets X and Y such that $X \cdot Y = H$, then there exist nonzero σ-finite G-left invariant (G-left-quasi-invariant) measures ν and ν' satisfying the following relations:
(1) ν' is a nonzero σ-finite G-left invariant (G-left-quasi-invariant) measure on G;
(2) $\nu' \supset \nu$;
(3) there exist two absolutely negligible sets X' and Y' such that $X' \cdot Y' = G$.

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Thank You for Your Attention