Ideal analytic sets

Łukasz Mazurkiewicz
Szymon Żeberski

Wrocław University of Science and Technology
lukasz.mazurkiewicz@pwr.edu.pl

Winter School in Abstract Analysis 2023
03.02.2023

Analytic complete sets

Definition

A set $A \subseteq X$ is called Σ^1_1-*complete* if A is analytic and for every Polish space Y and every analytic $B \subseteq Y$ there is a map $f : Y \to X$ satisfying $f^{-1}[A] = B$.

Analytic complete sets

Definition

A set $A \subseteq X$ is called Σ^1_1-*complete* if A is analytic and for every Polish space Y and every analytic $B \subseteq Y$ there is a map $f : Y \to X$ satisfying $f^{-1}[A] = B$.

Definition

Let $A \subseteq X$, $B \subseteq Y$. We say, that B is *Borel reducible* to A if there exists a Borel map $f : Y \to X$ such that $f^{-1}[A] = B$.

Analytic complete sets

Definition

A set $A \subseteq X$ is called Σ^1_1-*complete* if A is analytic and for every Polish space Y and every analytic $B \subseteq Y$ there is a map $f : Y \to X$ satisfying $f^{-1}[A] = B$.

Definition

Let $A \subseteq X$, $B \subseteq Y$. We say, that B is *Borel reducible* to A if there exists a Borel map $f : Y \to X$ such that $f^{-1}[A] = B$.

Fact

If analytic set B is Borel reducible to A and B is Σ^1_1-complete, then A is also Σ^1_1 complete.

Hindman ideal

For $B \subseteq \omega$ we write

$$FS(B) = \left\{ \sum_{n \in F} n : F \subseteq B, 0 < |F| < \omega \right\}.$$

Definition

A set $A \subseteq \omega$ is called an *IP-set* if there is an infinite $B \subseteq A$ such that $FS(B) \subseteq A$.

Hindman ideal

For $B \subseteq \omega$ we write

$$FS(B) = \left\{ \sum_{n \in F} n : F \subseteq B, 0 < |F| < \omega \right\}.$$

Definition

A set $A \subseteq \omega$ is called an *IP-set* if there is an infinite $B \subseteq A$ such that $FS(B) \subseteq A$.

Fact

Let $\mathcal{H}$ denote the family of all non-IP-sets. $\mathcal{H}$ is an ideal, called Hindman ideal.

Hindman ideal

For $B \subseteq \omega$ we write

$$FS(B) = \left\{ \sum_{n \in F} n : F \subseteq B, 0 < |F| < \omega \right\}.$$

Definition

A set $A \subseteq \omega$ is called an *IP-set* if there is an infinite $B \subseteq A$ such that $FS(B) \subseteq A$.

Fact

Let $\mathcal{H}$ denote the family of all non-IP-sets. $\mathcal{H}$ is an ideal, called Hindman ideal.

Fact

$\mathcal{H}$ is a coanalytic subset of $2^{\omega^{<\omega}}$.

Hindman ideal

Theorem

$\mathcal{H}$ is $\mathbf{\Pi}^1_1$-complete.

Hindman ideal

Theorem

$\mathcal{H}$ is $\mathbf{\Pi}^1_1$-complete.

Proof.

Let us fix an injection $\alpha : \omega^{<\omega} \to \{2^{2n} : n \in \omega\}$ satisfying

$$s \preceq t \Rightarrow \alpha(s) \leq \alpha(t).$$

Hindman ideal

Theorem

$\mathcal{H}$ is $\mathbf{\Pi}^1_1$-complete.

Proof.

Let us fix an injection $\alpha : \omega^{<\omega} \to \{2^{2n} : n \in \omega\}$ satisfying

$$s \preceq t \Rightarrow \alpha(s) \leq \alpha(t).$$

Define reduction $f : Tree_\omega \to P(\omega)$ with formula

$$f(T) = \bigcup_{s \in T} FS(\{\alpha(s \restriction k) : k \leq |s|\}).$$

□

Ideal $\mathcal{D}$

Analogously as before, for $B \subseteq \omega$

$$D(B) = \{b - a : b > a, a, b, \in B\} .$$

Definition

A set $A \subseteq \omega$ is called a *D-set* if there is an infinite $B \subseteq A$ such that $FS(B) \subseteq A$.

Ideal $\mathcal{D}$

Analogously as before, for $B \subseteq \omega$

$$D(B) = \{b - a : b > a, a, b, \in B\} .$$

Definition

A set $A \subseteq \omega$ is called a *D-set* if there is an infinite $B \subseteq A$ such that $FS(B) \subseteq A$.

Fact

Let $\mathcal{D}$ denote the family of all non-D-sets. $\mathcal{D}$ is a proper subideal of $\mathcal{H}$.

Ideal $\mathcal{D}$

Analogously as before, for $B \subseteq \omega$

$$D(B) = \{b - a : b > a, a, b, \in B\}.$$

Definition

A set $A \subseteq \omega$ is called a *D-set* if there is an infinite $B \subseteq A$ such that $FS(B) \subseteq A$.

Fact

Let $\mathcal{D}$ denote the family of all non-D-sets. $\mathcal{D}$ is a proper subideal of $\mathcal{H}$.

Fact

$\mathcal{D}$ is a coanalytic subset of $2^{\omega^{<\omega}}$.

Ideal $\mathcal{D}$

Theorem

$\mathcal{D}$ is $\mathbf{\Pi}^1_1$-complete.

Ideal $\mathcal{D}$

Theorem

$\mathcal{D}$ is $\mathbf{\Pi}^1_1$-complete.

Proof.

Let us fix an injection $\alpha : \omega^{<\omega} \to \{2^n : n \in \omega\}$ satisfying

$$s \preceq t \Rightarrow \alpha(s) \leq \alpha(t).$$

Ideal $\mathcal{D}$

Theorem

$\mathcal{D}$ is $\mathbf{\Pi}^1_1$-complete.

Proof.

Let us fix an injection $\alpha : \omega^{<\omega} \to \{2^n : n \in \omega\}$ satisfying

$$s \preceq t \Rightarrow \alpha(s) \leq \alpha(t).$$

Define reduction $f : Tree_\omega \to P(\omega)$ with formula

$$f(T) = \bigcup_{s \in T} D(\{\alpha(s \restriction k) : k \leq |s|\}).$$

□

Silver trees

Theorem

The set of all trees containing a Silver tree, i.e.

$$\left\{ T \in \mathit{Tree}_2 : \exists x \in 2^\omega \exists A \in [\omega]^\omega \forall s \in 2^{<\omega} (s = x \restriction |s| \Rightarrow s \in T) \right\},$$

is Σ^1_1-complete.

Silver trees

Theorem

The set of all trees containing a Silver tree, i.e.

$$\left\{T \in Tree_2 : \exists x \in 2^\omega \exists A \in [\omega]^\omega \forall s \in 2^{<\omega} (s = x \upharpoonright |s| \Rightarrow s \in T)\right\},$$

is Σ^1_1*-complete.*

Proof.

For $a = (a_0 a_1 a_2 \dots a_k)_2 \in \omega$, $s = (s_0 s_1 \dots s_l) \in 2^{<\omega}$ denote

$$\tilde{a} = (a_0 a_0 a_1 a_1 \dots a_k a_k)$$

$$\beta(s) = \{01\tilde{s}_0 01\underline{i_0} 01\tilde{s}_1 01\underline{i_1} \dots 01\tilde{s}_l 01\underline{i_l} : i_0, i_1, \dots \in \{0,1\}\}$$

Now define reduction $f : Tree_\omega \to Tree_2$ with formula

$$"f(T) = \bigcup_{s \in T} \beta(s)"$$

Silver trees - alternative approach

Theorem (Kechris-Louveau-Woodin)

Let $\mathcal{I} \subseteq K(X)$ be a coanalytic σ-ideal of compact sets. Then $\mathcal{I}$ is either G_δ or $\mathbf{\Pi}^1_1$-complete.

Silver trees - alternative approach

Theorem (Kechris-Louveau-Woodin)

Let $\mathcal{I} \subseteq K(X)$ be a coanalytic σ-ideal of compact sets. Then $\mathcal{I}$ is either G_δ or $\mathbf{\Pi}^1_1$-complete.

Fact

Family of all trees containing a Silver tree is $\mathbf{\Sigma}^1_1$-complete iff family of all bodies of trees containing a Silver tree is $\mathbf{\Sigma}^1_1$-complete.

Proof.

One way take $f(T) = [T]$, the other way $f(A) = \{\sigma \restriction k : \sigma \in A, k \in \omega\}$. □

Silver trees - alternative approach

Theorem (Kechris-Louveau-Woodin)

Let $\mathcal{I} \subseteq K(X)$ be a coanalytic σ-ideal of compact sets. Then $\mathcal{I}$ is either G_δ or $\mathbf{\Pi}^1_1$-complete.

Fact

Family of all trees containing a Silver tree is $\mathbf{\Sigma}^1_1$-complete iff family of all bodies of trees containing a Silver tree is $\mathbf{\Sigma}^1_1$-complete.

Proof.

One way take $f(T) = [T]$, the other way $f(A) = \{\sigma \restriction k : \sigma \in A, k \in \omega\}$. □

Using above for $X = 2^\omega$ and $\mathcal{I}$=compact sets, which do not contain a body of Silver tree, one gets result from the previous slide.

References

Rafał Filipów (2013)
On Hindman spaces and the Bolzano-Weierstrass property
Topology Appl. 160, no. 15

Alexander S. Kechris (1995)
Classical Descriptive Set Theory
Springer-Verlag New York, Inc.

Thank You for attention