# Square inside

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Mycielski J., Algebraic independence and measure, Fundamenta Mathematicae 61 (1967), 165-169.

**Theorem (Mycielski)**

*For every comeager or conull set* $G \subseteq [0,1]^2$ *there exists a perfect set* $P \subseteq [0,1]$ *such that* $P \times P \subseteq G \cup \Delta$.

$\Delta = \{(x,x) : x \in [0,1]\}$.

Let $A \in \{2, \omega\}$ and let $T \subseteq A^{<\omega}$ be a tree on $A$, i.e. for each $\sigma \in T$ we have $\sigma \restriction n \in T$ for all natural $n$. Body of a tree $T$ is the set

$$[T] = \{x \in A^\omega : (\forall n \in \omega)(x \restriction n \in T)\}$$

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Perfect sets = bodies of perfect trees.
The goal: to switch to $2^\omega$ and $\omega^\omega$ and to replace a perfect set with a body of some tree.

Denote

$$\mathrm{split}(T) = \{\sigma \in T : (\exists n, k \in A)(n \neq k \ \&\ \sigma^\frown n, \sigma^\frown k \in T)\}.$$

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## Definition

We call a tree $T \subseteq A^{<\omega}$

- a perfect or Sacks tree, if for each $\sigma \in T$ there is $\tau \in T$ such that $\sigma \subseteq \tau$ and $\tau^\frown n, \tau^\frown k \in T$ for distinct $n, k \in A$;
- uniformly perfect, if it is perfect and for each $n \in \omega$ either $A^n \cap T \subseteq \mathrm{split}(T)$ or $\mathrm{split}(T) \cap A^n = \emptyset$;
- a Silver tree, if it is perfect and for all $\sigma, \tau \in T$ with $|\sigma| = |\tau|$ we have $\sigma^\frown n \in T \Leftrightarrow \tau^\frown n \in T$ for all $n \in A$;
- a splitting tree ($A = 2$) if for every $\sigma \in T$ there is $N \in \omega$ such that for each $n > N$ and $i \in \{0, 1\}$ there are $\tau_0, \tau_1 \in T$ such that $\sigma \subseteq \tau_i{}^\frown i \in T$;

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- a Miller tree ($A = \omega$), if for each $\sigma \in T$ there exists $\tau \in T$ such that $\sigma \subseteq \tau$ and for infinitely many $n \in A$ we have $\tau^\frown n \in T$;
- a Laver tree ($A = \omega$), if there is $\sigma \in T$ such that for each $\tau \in T$ satisfying $\sigma \subseteq \tau$ there are infinitely many $n \in A$ with $\tau^\frown n \in T$;

# Measure case - Miller trees

**Proposition**

*Let $\mu$ be a strictly positive probabilistic measure on $\omega^\omega$. Then there exists an $F_\sigma$ set $F$ of measure $1$ such that $[T] \not\subseteq F$ for every Miller tree $T$.*

# Measure case - Sacks trees

**Theorem**

*Let $F$ be a subset of $2^\omega \times 2^\omega$ of full measure. Then there exists a uniformly perfect tree $T \subseteq 2^{<\omega}$ satisfying $[T] \times [T] \subseteq F \cup \Delta$.*

# Measure case - Silver trees

## Definition

$A \subseteq 2^\omega$ is a small set if there is a partition $\mathcal{A}$ of $\omega$ into finite sets and a collection $(J_a)_{a\in\mathcal{A}}$ such that $J_a \subseteq 2^a$, $\sum_{a\in\mathcal{A}} \frac{|J_a|}{2^{|a|}} < \infty$ and

$$A \subseteq \{x \in 2^\omega : \ (\exists^\infty a \in \mathcal{A})(x \restriction a \in J_a)\}.$$

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## Proposition

*There exist a small set* $A \subseteq 2^\omega \times 2^\omega$ *such that* $(A \cap [T] \times [T]) \setminus \Delta \neq \emptyset$ *for any Silver tree* $T \subseteq 2^{<\omega}$.

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*There exist a small set $A \subseteq 2^\omega \times 2^\omega$ such that $(A \cap [T] \times [T]) \setminus \Delta \neq \emptyset$ for any Silver tree $T \subseteq 2^{<\omega}$.*

## Proposition

*Every closed subset of $2^\omega$ of positive Lebesgue measure contains a Silver tree.*

# Category case - Laver trees

**Proposition**

*There exists a dense $G_\delta$ set $G \subseteq \omega^\omega$ such that $[T] \not\subseteq G$ for every Laver tree $T$.*

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## Proof.

$G = \{x \in \omega^\omega : (\exists^\infty n \in \omega)(x(n) = 0)\}$. □

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**Lemma**

*For every Silver tree $T$ there exists a Silver tree $T' \subseteq T$ that splits and rests.*

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*There exists an open dense set $U \subseteq \omega^\omega \times \omega^\omega$ such that $[T] \times [T] \not\subseteq U \cup \Delta$ for any Silver tree $T$.*

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**Proposition**

*There exists an open dense set $U \subseteq \omega^\omega \times \omega^\omega$ such that $[T] \times [T] \nsubseteq U \cup \Delta$ for any Silver tree $T$.*

**Proof.**

Let $\text{supp} : \mathbb{Q} \to \omega$ be given by $\text{supp}(\mathbb{0}) = 0$ and $\text{supp}(q) = \max\{n \in \omega : q(n) \neq 0\}$ for $q \neq \mathbb{0}$. Let $\{(q_1^n, q_2^n) : n \in \omega\}$ be an enumeration of pairs rationals for which $\text{supp}(q_1^n) = \text{supp}(q_2^n)$.

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$$U = \bigcup_{n \in \omega} [(q_1^n \restriction (\mathrm{supp}(q_1^n)))^\frown(0,0)] \times [(q_2^n \restriction (\mathrm{supp}(q_1^n)))^\frown(1,1)].$$

□

# Category case - Miller trees

**Theorem**

*For every comeager set $G$ of $\omega^\omega \times \omega^\omega$ there exists a Miller tree $T_M \subseteq \omega^{<\omega}$ and a uniformly perfect tree $T_P \subseteq T_M$ such that $[T_P] \times [T_M] \subseteq G \cup \Delta$.*

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**Remark**

*A Miller tree cannot be replaced with a uniformly perfect Miller tree.*

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**Remark**

*A Miller tree cannot be replaced with a uniformly perfect Miller tree.*

**Theorem**

*There exists a dense $G_\delta$ set $G \subseteq \omega^\omega \times \omega^\omega$ such that $[T_1] \times [T_2] \not\subseteq G \cup \Delta$ for any Miller trees $T_1, T_2$.*

See also

S. Solecki, O. Spinas, Dominating and unbounded free sets, Journal of Symbolic Logic 64 (1999), 75-80.

# Category case - splitting trees

Denote

$$\Delta_n = \{(x_1, x_2, ..., x_n) \in (2^\omega)^n : (\exists i, j \in \{1, 2, ..., n\})(i \neq j \ \& \ x_i = x_j)\}.$$

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**Theorem**

*Let $(G_n : n > 0)$ be a sequence of comeager sets, $G_n \subseteq (2^\omega)^n$ for each $n > 0$. Then there exists a uniformly perfect splitting tree $T \subseteq 2^{<\omega}$ such that $[T]^n \subseteq G_n \cup \Delta_n$.*

# Category case - splitting trees

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**Sketch of a proof.**

On a blackboard if time allows. □

# Application

**Corollary**

*There is a uniformly perfect splitting tree* $T \subseteq 2^{<\omega}$ *such that* $|[T] \cap (x + [T])| \leq 1$ *for* $x \neq \mathbb{0}$.

# Thank you for your attention!

Michalski M., Rałowski R. Żeberski Sz., Mycielski among trees, Mathematical Logic Quarterly 67 (3) (2021), 271-281.