Winter School in Abstract Analysis 2023

Colorings of Abelian groups

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Joint work with Assaf Rinot

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Outline

Additive Ramsey Theory

The S-principle

More colors, higher dimensions

Motivation

Let us recall Ramsey theorem from 1930. For every partition $[\mathbb{N}]^2 = A \uplus B$ there exists an infinite set $X \subseteq \mathbb{N}$ such that $[X]^2 \subseteq A$ or $[X]^2 \subseteq B$.

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But, Sierpiński proved in 1933 that if we consider the set $\mathbb{R}$, there exists a partition $[\mathbb{R}]^2 = A \uplus B$ such that, for every $X \subseteq \mathbb{R}$ uncountable, $[X]^2$ is not contained in A nor B.

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But, $\mathbb{N}$ and $\mathbb{R}$ are also Abelian semi-groups.
What if we take into consideration the algebraic structure on those sets?

Historical Background

- Hindman's proved in 1974 that considering $(\mathbb{N}, +)$ for every partition into two cells $\mathbb{N} = A \uplus B$ there exists an infinite set $X \subseteq \mathbb{N}$ such that, the set of all finite sums of X ($\mathrm{FS}(X)$) is contained in either A or B.
- On the other hand, Komjáth proved in 2016 that $(\mathbb{R}, +)$ admits the opposite property.
  Namely, there exists a partition $[\mathbb{R}]^2 = A \uplus B$ such that, for every uncountable set $X \subseteq \mathbb{R}$ the set of all the sums of two elements from X ($\mathrm{FS}_2(X)$) is not contained in A nor B.

Generalization

Consider the following "Ramsey-type" problem:
For $\theta \leq \lambda \leq \kappa$ infinite cardinals. Given an Abelian (semi) group $(G, +)$ of size κ, for all colorings $c : G \to \theta$, there exists a set $X \subseteq G$ of size λ such that the set of all finite sums of elements from X is monochromatic.

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We shall abbreviate this sentence by:

$$G \to (\lambda)_\theta^{\mathsf{FS}}$$

and if we restrict ourselves only to sums of two elements,

$$G \to (\lambda)_\theta^{\mathsf{FS}_2}.$$

Main result

Theorem (Special case)

Under $\neg$*(CH). For every Abelian group of size* $\aleph_2$ *there exists a coloring* $c : G \to \omega$ *such that, for every subset* X *of* G *of size* $\aleph_1$ *and color* $n < \omega$*, we may find* $x, y, z \in X$ *for whom* $c(x + y + z) = n$*.*

i.e. $G \nrightarrow [\omega_1]^{\mathrm{FS}_3}_{\omega}$ *for all Abelian groups* G *of size* $\aleph_2$*.*

Notation

We commence with brief recall of the "Classical Ramsey-theory" definitions.

Definition

- $\kappa \nrightarrow [\lambda]^2_\theta$ asserts the existence of a coloring $c : [\kappa]^2 \rightarrow \theta$ such that, for every $A \in [\kappa]^\lambda$, $c``[A]^2 = \theta$;
- $\kappa \nrightarrow [\lambda; \lambda]^2_\theta$ asserts the existence of a coloring $c : [\kappa]^2 \rightarrow \theta$ such that, for all $A, B \in [\kappa]^\lambda$, $c[A \circledast B] = \theta$.

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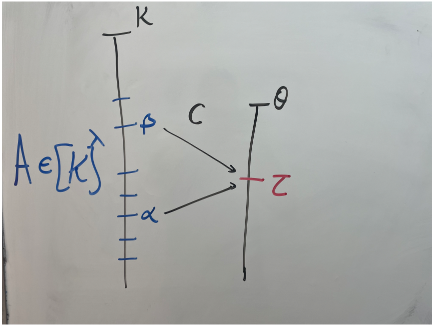
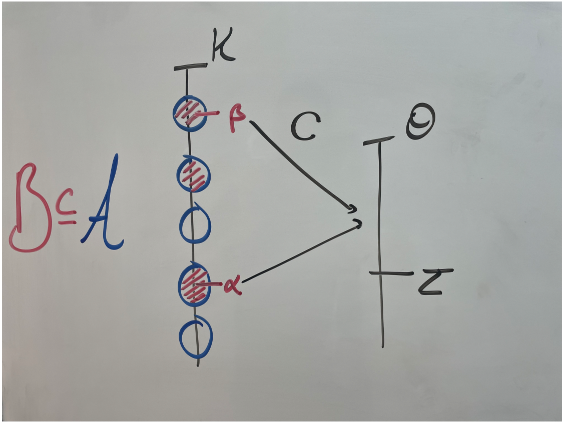


Notation

Definition (Lambie-Hanson and Rinot, 2018)

$\mathrm{U}(\kappa, \mu, \theta, \chi)$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ such that for every $\sigma < \chi$, every κ-sized pairwise disjoint subfamily $\mathcal{A} \subseteq [\kappa]^\sigma$, and every $\tau < \theta$, there exists $\mathcal{B} \in [\mathcal{A}]^\mu$ such that $\min(c[a \times b]) > \tau$ for all $(a, b) \in [\mathcal{B}]^2$.



Strong Failures

- Fernández-Bretón and Rinot's theorem from 2017 showed that $G \nrightarrow [\omega_1]^{\mathsf{FS}}_{\omega}$ for every uncountable Abelian group G.
  i.e. there exists a coloring $c : G \to \omega$ such that for all $X \subseteq G$ uncountable, $c``\mathsf{FS}(X) = \omega$.
- In the same paper, Fernández-Bretón and Rinot showed that for class many infinite cardinals λ, $G \nrightarrow [\lambda]^{\mathsf{FS}_2}_{\omega}$ holds for every abelian group G of size λ.
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Reduction

Fact (Representing Abelian groups as direct sum)

Suppose that G is an infinite Abelian group. Denote by κ the size of G. Then, there exists a sequence of countable divisible groups $\langle G_\alpha \mid \alpha < \kappa \rangle$ such that G embeds in $\bigoplus_{\alpha<\kappa} G_\alpha$.

Reduction

Fact (Representing Abelian groups as direct sum)

Suppose that G is an Abelian group of size κ. Then, there exists a sequence of countable divisible groups $\langle G_\alpha \mid \alpha < \kappa \rangle$ such that G embeds in $\bigoplus_{\alpha<\kappa} G_\alpha$.

Thus, if we replace every element $x \in G$ by $\mathrm{supp}(x)$ our problem may be translated as follows,

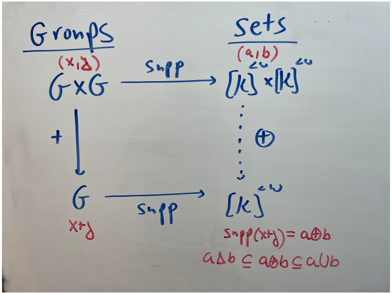
Definition (The S-principle)

$S_n(\kappa, \lambda, \theta)$ asserts the existence of a coloring $f : [\kappa]^{<\omega} \to \theta$ such that, for every $\mathcal{X} \subseteq [\kappa]^{<\omega}$ of size λ and a color $\tau < \theta$, there exist $\{a_j \mid j < n\} \in [\mathcal{X}]^n$ such that, for every z satisfying

$$(a_0) \triangle \left(\bigcup_{0<j<n} a_j \right) \subseteq z \subseteq \bigcup_{j<n} a_j,$$

$f(z) = \tau$.

Reduction



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Background.

Lemma 1. (Fernández-Bretón-Rinot, 2017)
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Lemma 2. (Fernández-Bretón-Rinot, 2017)
For every successor κ: $\kappa \nrightarrow [\kappa]^2_\theta$ holds iff $S_2(\kappa, \kappa, \theta)$ holds.

Extraction principle

Extraction principles are maps that help detecting Δ-systems within a big family of finite sets.

Definition

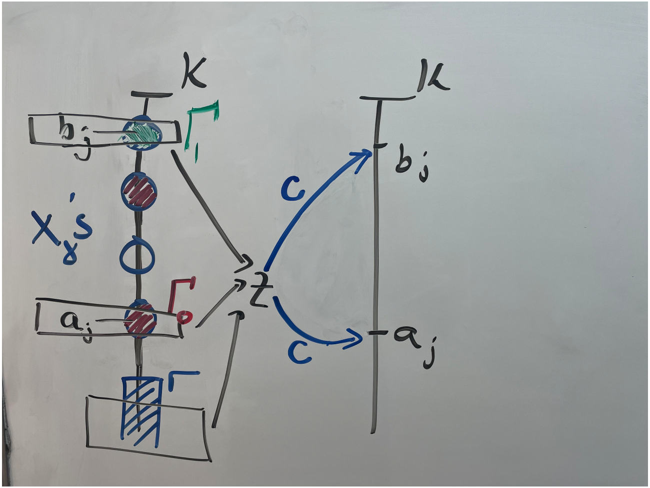
$\mathrm{Extract}_2(\kappa, \lambda, \mu, \chi)$ asserts the existence of a map $e : [\kappa]^{<\omega} \to [\kappa]^2$ such that:

1. for every $z \in [\kappa]^{<\omega}$ of size ≥ 2, $e(z) \in [z]^2$;
2. for every sequence $\langle x_\gamma \mid \gamma < \lambda \rangle$ of subsets of κ, every $r \in [\kappa]^{<\mu}$, and every nonzero $\sigma < \chi$ such that:
   - 2.1 for every $(\gamma, \gamma') \in [\lambda]^2$, $x_\gamma \cap x_{\gamma'} \subseteq r$;
   - 2.2 for every $\gamma < \lambda$, $y_\gamma := x_\gamma \setminus r$ has order-type σ,

   there exist $j < \sigma$ and cofinal subsets Γ_0, Γ_1 of λ satisfying the following. For every $(\gamma, \gamma') \in (\Gamma_0 \circledast \Gamma_1) \cup (\Gamma_1 \circledast \Gamma_0)$, for every $z \in [x_\gamma \cup x_{\gamma'}]^{<\omega}$ covering $\{y_\gamma(j), y_{\gamma'}(j)\}$, we have

$$e(z) = (y_\gamma(j), y_{\gamma'}(j)).$$

Extraction principle



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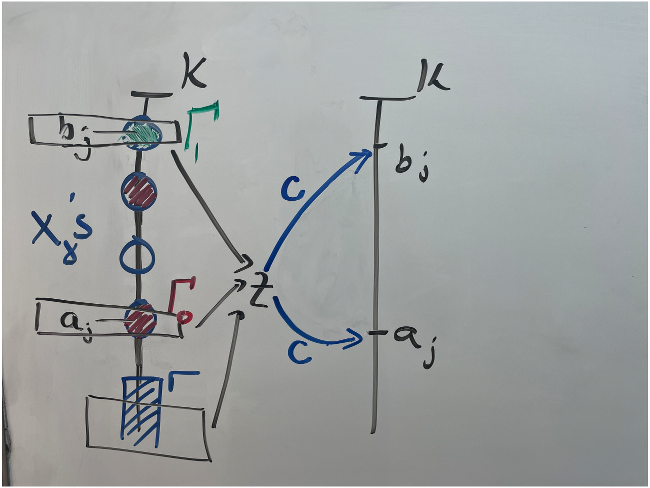

Example

Suppose that $\mathrm{U}(\kappa, \kappa, \omega, \omega)$ holds for a regular uncountable κ and an infinite $\theta < \kappa$. Then, $\mathrm{Extract}_2(\kappa, \kappa, \omega, \omega)$ holds.

Proof of the example

Fix $c : [\kappa]^2 \rightarrow \omega$ witnessing $\mathrm{U}(\kappa, \kappa, \omega, \omega)$. Define a coloring $d : [\kappa]^{<\omega} \rightarrow \theta$, as follows. For $z \in [\kappa]^{<2}$, just let $d(z) := (0, 1)$. Next, for $z \in [\kappa]^{<\omega}$ of size ≥ 2, first let $\langle \alpha_i \mid i < |z| \rangle$ denote the increasing enumeration of z. Then set

$$j_z := \min\{j < |z|-1 \mid c(\alpha_j, \alpha_{j+1}) = \max\{c(\alpha_i, \alpha_{i+1}) \mid i < |z|-1\}\},$$

and let $d(z) := (\alpha_{j_z}, \alpha_{|z|})$.

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Lemma (General technique)

Assume $\mathrm{Extract}_2(\kappa, \lambda, \omega, \omega)$ *and* $\kappa \nrightarrow [\lambda, \lambda]^2_\theta$ *hold, then* $S_2(\kappa, \lambda, \theta)$ *holds.*

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Lemma

For λ *regular uncountable, if* $\kappa > 2^{<\lambda}$ *then* $\mathrm{Extract}_2(\kappa, \lambda, 2, 2)$ *fails.*

An observation

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Definition

$\kappa \overset{\text{sup}}{\nrightarrow} [\lambda, \lambda]^2_\theta$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ such that for all $\tau < \theta$ and disjoint $A, B \in \mathcal{P}(\kappa)$ satisfying the two:

1. $\text{otp}(A) = \text{otp}(B) = \lambda$,
2. $\sup(A) = \sup(B)$,

there is $(\alpha, \beta) \in [A \cup B]^n \setminus ([A]^n \cup [B]^n)$ with $c(\alpha, \beta) = \tau$.

The case $\kappa = \lambda$

In case $\kappa = \lambda$ the two relations are equivalent and by similar argument as the example before $\text{Extract}_2(\kappa, \kappa, \omega, \omega)$ holds.

Lemma

Suppose that $\kappa \nrightarrow [\kappa; \kappa]^2_\theta$ holds for a regular uncountable κ and an infinite $\theta \leq \kappa$. Then $S_2(\kappa, \kappa, \theta)$ holds, as well.

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Lemma

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Question

What can be said on $S_2(\kappa, \lambda, \theta)$ when $\lambda < \kappa$?

The case $\lambda < \kappa$

Theorem

If there exists a weak μ-Kurepa tree with κ branches, then $S_2(\kappa, \lambda, 2)$ holds, for $\lambda := \mu^+$.

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The existence of weak μ-Kurepa tree with κ branches gives us:

- $\text{Extract}_2(\kappa, \lambda, \mu, \omega)$ for every regular cardinal $\lambda \in (\mu, \kappa]$;
- a coloring witnessing $\kappa \overset{\text{sup}}{\nrightarrow} [\lambda, \lambda]^2_2$.

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Corollary

For every infinite cardinal $\lambda = 2^{<\lambda}$, $S_2(2^\lambda, \lambda^+, 2)$ holds.

Corollary (Komjáth, 2016)

$\mathbb{R} \nrightarrow [\omega_1]_2^{\mathrm{FS}_n}$ *for any $n < \omega$;*

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Corollary (Komjáth, 2020)

There exists $c : \mathbb{R} \to 2$ such that, for every $i < 2$ and $X \subseteq \mathbb{R}$ of size $\aleph_1$, there exist $x \neq y \in X$ with $c(|x - y|) = i$.

Outline

This section is dedicated to give a brief overview of the tools we used to prove the main result.

As before

Lemma

Suppose that there exists a weak μ-Kurepa tree with at least κ many branches. Then $\text{Extract}_3(\kappa, \lambda, \mu, \omega)$ *for every regular cardinal* $\lambda \in (\mu, \kappa]$.

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But what about the appropriate coloring?

The coloring

Definition

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there is $\vec{x} \in [A \cup B]^n \setminus ([A]^n \cup [B]^n)$ with $c(\vec{x}) = \tau$.

Lemma

Suppose that:

- $2 \leq n < \omega$;
- $\theta \leq \lambda \leq \kappa$ *are cardinals with* λ *regular and uncountable;*
- $\kappa \overset{\text{sup}}{\nrightarrow} [\lambda, \lambda]^n_\theta$;
- $\text{Extract}_n(\kappa, \lambda, \omega, \omega)$ *holds.*

Then $S_n(\kappa, \lambda, \theta)$ *holds.*

The coloring: maximal number of colors

Theorem

The following are equivalent:

1. $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$ *fails;*
2. *There exist a coloring* $c : [\omega_2]^3 \to \omega_1$ *with the property that for all disjoint* $A, B \subseteq \omega_2$ *of order-type* ω_1 *such that* $\sup(A) = \sup(B)$, *for every color* $\tau < \omega_1$, *there is* $(\alpha, \beta, \gamma) \in [A \cup B]^3 \setminus ([A]^3 \cup [B]^3)$ *such that* $c(\alpha, \beta, \gamma) = \tau$.
   i.e. $\omega_2 \overset{\sup}{\nrightarrow} [\omega_1, \omega_1]^3_{\omega_1}$

The coloring: countably many colors

Theorem

Suppose that $\lambda = \mu^+$ *for an infinite cardinal* $\mu = \mu^{<\mu}$.
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- In the first case, we get a similar situation as in the maximal color case. i.e. Chang's conjecture fails.
- In the other case, we use a lifting up of the oscillation map.

Some open questions regarding the Extract

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Recall, $\text{Extract}_2(\aleph_2, \aleph_1, \aleph_0, \aleph_0)$ holds iff CH fails.

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Is there a model of ZFC such that, $\text{Extract}_3(\kappa, \lambda, \omega, \omega)$ *holds but* $\text{Extract}_2(\kappa, \lambda, \omega, \omega)$ *fails?*

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Question

Is $\mathsf{U}(\kappa, \lambda, \ldots)$ *imply* $\mathsf{Extract}_2(\kappa, \lambda, \ldots)$*?*

The paper is available in: http://p.assafrinot.com/57

Questions?