On κ-Corson compact spaces and reated classes of compact spaces

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Equivalently, a compact space K is an Eberlein compactum if K can be embedded in the following subspace of the product $\mathbb{R}^\Gamma$:

$$c_0(\Gamma) = \{x \in \mathbb{R}^\Gamma : \text{ for every } \varepsilon > 0 \text{ the set } \{\gamma : |x(\gamma)| > \varepsilon\} \text{ is finite}\},$$

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Continuous images, closed subspaces, countable products of Eberlein compacta are Eberlein compact spaces.

A compact space K is Corson compact if, for some set Γ, K is homeomorphic to a subset of the Σ-product of real lines

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Clearly, the class of Corson compact spaces contains all Eberlein compacta.
Let κ be an infinite cardinal number. A compact space K is κ-Corson compact if, for some set Γ, K is homeomorphic to a subset of the Σ_κ-product of real lines

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Obviously, the class of Corson compact spaces coincides with the class of ω_1-Corson compact spaces.

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$$\sigma(X_\gamma, a_\gamma, \Gamma) = \{(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X_\gamma : |\{\gamma \in \Gamma : x_\gamma \neq a_\gamma\}| < \omega\}.$$

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If $X_\gamma = I = [0,1]$ and $a_\gamma = 0$, for all $\gamma \in \Gamma$, then we denote the σ-product $\sigma(X_\gamma, a_\gamma, \Gamma)$ by $\sigma(I, \Gamma)$.

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If $X_\gamma = I^\omega$ and $a_\gamma = (0, 0, \dots)$, for all $\gamma \in \Gamma$, then we denote the σ-product $\sigma(X_\gamma, a_\gamma, \Gamma)$ by $\sigma(I^\omega, \Gamma)$.

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For $\kappa = \omega$, $\Sigma_\kappa(\mathbb{R}^\Gamma) = \sigma(\mathbb{R}, \Gamma)$.

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Proposition

For a compact space K we have

(a) *K is ω-Corson if and only if it can be embedded into some σ-product of metrizable finitely dimensional compacta if and only if it can be embedded into the σ-product $\sigma(I, \Gamma)$ for some set Γ.*

(b) *K is NY compact if and only if it can be embedded into the σ-product $\sigma(I^\omega, \Gamma)$ for some set Γ.*

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Given a family $\mathcal{U}$ of subsets of a space X, a point $x \in X$, and an infinite cardinal κ, we write $ord(x, \mathcal{U}) < \kappa$ if $|\{U \in \mathcal{U} : x \in U\}| < \kappa$.

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We say that $\mathcal{U}$ is point-finite if $ord(x, \mathcal{U}) < \omega$ for all $x \in X$.

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We say that $\mathcal{U}$ is point-finite if $ord(x,\mathcal{U}) < \omega$ for all $x \in X$.

Proposition

Let κ be an uncountable cardinal number. For a compact space K, the following conditions are equivalent:

1. *K is κ-Corson;*
2. *There exists a family $\mathcal{U}$ consisting of cozero subsets of K which is T_0-separating, and $ord(x,\mathcal{U}) < \kappa$ for all $x \in K$.*

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Proposition (Marciszewski, Plebanek, Z.)

For a compact space K, the following conditions are equivalent:

- (a) *There exists a T_0-separating, point-finite family $\mathcal{U}$ consisting of cozero subsets of K;*
- (b) *K is a scattered Eberlein compact space.*

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All scattered Eberlein compacta are ω-Corson.

A family $\mathcal{A}$ of subsets of a space X is closure preserving if, for any subfamily $\mathcal{A}' \subseteq \mathcal{A}$, we have

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Theorem (Z., Marciszewski, Plebanek)

For a compact space K, the following conditions are equivalent:

1. *K is ω-Corson;*
2. *K has a closure preserving cover consisting of finite dimensional metrizable compacta;*
3. *K is hereditarily metacompact and each nonempty subspace A of K contains a nonempty relatively open separable, metrizable, finite dimensional subspace U.*

Theorem (Z., Marciszewski, Plebanek)

For a compact space K, the following conditions are equivalent:

- (a) *K belongs to the class $\mathcal{N}\mathcal{Y}$;*
- (b) *There exists a T_0-separating family $\mathcal{U} = \bigcup\{\mathcal{U}_\gamma : \gamma \in \Gamma\}$ consisting of cozero subsets of K, where each $\mathcal{U}_\gamma$ is a countable and the family $\{\bigcup\mathcal{U}_\gamma : \gamma \in \Gamma\}$ is point-finite;*
- (c) *K has a closure preserving cover consisting of metrizable compacta;*
- (d) *K is hereditarily metacompact and each nonempty subspace A of K contains a nonempty relatively open subspace U of countable weight.*

Theorem (Z., Marciszewski, Plebanek)

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The equivalence of conditions (a-c) was proved by Nakhmanson and Yakovlev.

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Corollary (Nakhmanson and Yakovlev)

The class $\mathcal{NY}$ is stable under continuous images

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For any sequence $(K_n)_{n\in\omega}$ of nonmetrizable Eberlein compacta, the product $\prod_{n\in\omega} K_n$ does not belong to $\mathcal{NY}$.

Theorem (Gruenhage)

For a compact space K, the following conditions are equivalent:

(a) *K is Eberlein compact;*

(b) *K^2 is hereditarily σ-metacompact;*

(c) *$K^2 \setminus \Delta$ is σ-metacompact.*

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Example (Marciszewski, Plebanek, Z.)

There exist a zero-dimensional Eberlein compact space K such that K^n is hereditarily metacompact for every $n \in \omega$, but K is not ω-Corson.

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Theorem (Marciszewski, Plebanek, Z.)

Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of nonmetrizable Eberlein compact spaces, then $\prod_{n\in\mathbb{N}} K_n$ is not hereditary metacompact.

The class of ω-Corson compact spaces is clearly stable under taking closed subspaces and finite products, but is not stable under taking continuous images, as the Hilbert cube is a continuous image of the Cantor set 2^ω.

Definition

For $\lambda, \kappa \in \mathit{Card}$, let $L_\kappa(\lambda) = \lambda \cup \{\infty\}$ where all $\alpha \in \lambda$ are discrete and basis neighbourhoods of ∞ are of the form $A \cup \{\infty\}$ where $A \subseteq \lambda$ and $|\lambda \setminus A| < \kappa$.

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Definition

Let $\mathcal{L}(\kappa)$ be the class of all continuous images of closed subspaces of $L_\kappa(\lambda)^\omega$ for $\lambda \in Card$.

Theorem (Bell, Marciszewski)

Let K be a compact space and let $\kappa \geq \omega$.
Space K is κ^+-Corson $\iff$ $C_P(K) \in \mathcal{L}_{\kappa^+}$.

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Let K be a compact space and let $\kappa > \omega$.
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Let K be a compact space and let $\kappa > \omega$ be a regular cardinal number.
Space K is κ-Corson $\iff$ $C_P(K) \in \mathcal{L}_\kappa$.