# On weaker types of continuity with a little help of ideals[1]

**Aleksandar Pavlović** Anika Njamcul

Department of Mathematics and Informatics,
Faculty of Sciences,
Novi Sad, Serbia

Winter School 2023

[1] This talk is supported by the Science Fund of the Republic of Serbia, Grant No. 7750027: Set-theoretic, model-theoretic and Ramsey- theoretic phenomena in mathematical structures: similarity and diversity – SMART

# Local function

$\langle X, \tau \rangle$ - topological space

$$x \in \mathrm{Cl}(A) \Leftrightarrow \text{ for each } U \in \tau(x) \; A \cap U \notin \{\emptyset\}$$

# Local function

$\langle X, \tau \rangle$ - topological space

$$x \in \mathrm{Cl}(A) \Leftrightarrow \text{ for each } U \in \tau(x)\ A \cap U \notin \{\emptyset\}$$

$\mathcal{I}$ - an ideal on $X$

$$x \in A^*_{(\tau,\mathcal{I})} \Leftrightarrow \text{ for each } U \in \tau(x)\ A \cap U \notin \mathcal{I}$$

$\langle X, \tau, \mathcal{I} \rangle$ - ideal topological space [Kuratowski 1933]

$A^*_{(\tau,\mathcal{I})}$ (briefly $A^*$) - **local function**

# Local function

For $\mathcal{I} = \{\emptyset\}$ we have that $A^*(\mathcal{I}, \tau) = \mathrm{Cl}(A)$.
For $\mathcal{I} = P(X)$ we have that $A^*(\mathcal{I}, \tau) = \emptyset$.
For $\mathcal{I} = Fin$ we have that $A^*(\mathcal{I}, \tau)$ is the set of $\omega$-accumulation points of $A$.
For $\mathcal{I} = \mathcal{I}_{count}$ we have that $A^*(\mathcal{I}, \tau)$ is the set of condensation points of $A$.

# Local function

For $\mathcal{I} = \{\emptyset\}$ we have that $A^*(\mathcal{I}, \tau) = \mathrm{Cl}(A)$.
For $\mathcal{I} = P(X)$ we have that $A^*(\mathcal{I}, \tau) = \emptyset$.
For $\mathcal{I} = Fin$ we have that $A^*(\mathcal{I}, \tau)$ is the set of $\omega$-accumulation points of $A$.
For $\mathcal{I} = \mathcal{I}_{count}$ we have that $A^*(\mathcal{I}, \tau)$ is the set of condensation points of $A$.

(1) $A \subseteq B \Rightarrow A^* \subseteq B^*$;
(2) $A^* = \mathrm{Cl}(A^*) \subseteq \mathrm{Cl}(A)$;
(3) $(A^*)^* \subseteq A^*$;
(4) $(A \cup B)^* = A^* \cup B^*$
(5) If $I \in \mathcal{I}$, then $(A \cup I)^* = A^* = (A \setminus I)^*$.

# Topology $\tau^*$

### Definition

$\mathrm{Cl}^*(A) = A \cup A^*$ is a Kuratowski closure operator, and therefore it generates a topology on $X$

$$\tau^*(\mathcal{I}) = \{A : \mathrm{Cl}^*(X \setminus A) = X \setminus A\}.$$

Set $A$ is closed in $\tau^*$ iff $A^* \subseteq A$.

$$\psi(A) = X \setminus (X \setminus A)^*$$

$$O \in \tau^* \Leftrightarrow O \subseteq \psi(O)$$

$$\tau \subseteq \tau^* = \tau^{**}$$

$$\beta(\mathcal{I}, \tau) = \{V \setminus I : V \in \tau, I \in \mathcal{I}\} \text{ is a basis for } \tau^*$$

# Topology $\tau^*$

For $\mathcal{I} = \{\emptyset\}$ we have that $\tau^*(\mathcal{I}) = \tau$.
For $\mathcal{I} = P(X)$ we have that $\tau^*(\mathcal{I}) = P(X)$.
If $\mathcal{I} \subseteq \mathcal{J}$ then $\tau^*(\mathcal{I}) \subseteq \tau^*(\mathcal{J})$.
If $Fin \subseteq \mathcal{I}$ then $\langle X, \tau^* \rangle$ is $T_1$ space.
If $\mathcal{I} = Fin$ , then $\tau^*_{ad}(\mathcal{I})$ is the cofinite topology on $X$.
If $\mathcal{I} = \mathcal{I}_{m0}$ - ideal of the sets of measure zero, then $\tau^*$-Borel sets are precisely the Lebesgue measurable sets. (Scheinberg 1971)
For $\mathcal{I} = \mathcal{I}_{nwd}$ then $A^* = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$ and $\tau^*(\mathcal{I}_{nwd}) = \tau^\alpha$. ($\alpha$-open sets, $A \subseteq \mathrm{Int}(Cl(Int(A)))$. (Njástad 1965)

# $\theta$-open sets

$$x \in \mathrm{Cl}_\theta(A) \Leftrightarrow \forall U \in \tau(x) \;\; \mathrm{Cl}(U) \cap A \notin \{\emptyset\}$$

$A$ is $\theta$-closed iff $\mathrm{Cl}_\theta(A) = A$

$A$ is $\theta$-open iff $X \setminus A$ is $\theta$-closed

Introduced by Veličko in 1968 in order to study H-closed spaces and H-sets.

Set is $A$ H-set (in Hausdorff space $X$) iff for each open cover $\{U_\alpha : \alpha < \kappa\}$ of $A$ exists finite subfamily $\{U_{\alpha_k} : k \leq n\}$ such that $A \subseteq \bigcup_{k \leq n} \mathrm{Cl}(U_{\alpha_k})$.

# $\theta$-open sets

$$x \in \mathrm{Cl}_\theta(A) \Leftrightarrow \forall U \in \tau(x)\ \ \mathrm{Cl}(U) \cap A \notin \{\emptyset\}$$

$A$ is $\theta$-closed iff $\mathrm{Cl}_\theta(A) = A$
$A$ is $\theta$-open iff $X \setminus A$ is $\theta$-closed
Introduced by Veličko in 1968 in order to study H-closed spaces and H-sets.

Set is $A$ H-set (in Hausdorff space $X$) iff for each open cover $\{U_\alpha : \alpha < \kappa\}$ of $A$ exists finite subfamily $\{U_{\alpha_k} : k \leq n\}$ such that $A \subseteq \bigcup_{k \leq n} \mathrm{Cl}(U_{\alpha_k})$.

$\theta$ open sets forms topology $\tau_\theta$ and $\tau_\theta \subseteq \tau$

Space is $T_3$ iff $\tau_\theta = \tau$

$$A \subseteq \mathrm{Cl}_\theta(A) \subseteq \mathrm{Cl}_{\tau_\theta}(A)$$

# Local closure function

$$x \in \Gamma_{(\tau,\mathcal{I})}(A) \Leftrightarrow \forall U \in \tau(x)\ \ \mathrm{Cl}(U) \cap A \notin \mathcal{I}$$

If $\mathcal{I} = \{\emptyset\}$ then $\Gamma(A) = \mathrm{Cl}_\theta(A)$
$\Gamma$ -local closure function
Introduced by Al-Omari and Noiri [1] in 2013.

$$\psi_\Gamma(A) = X \setminus \Gamma(X \setminus A)$$

Topology $\sigma$ is defined by $\psi_\Gamma$:

$$A \in \sigma \Leftrightarrow A \subseteq \psi_\Gamma(A).$$

$F$ is a closed set $\sigma$ iff $\Gamma(F) \subseteq F$.
$\tau_\theta \subseteq \sigma$
If $\mathcal{I} = \{\emptyset\}$, then $\tau_\theta = \sigma$.

### Definition

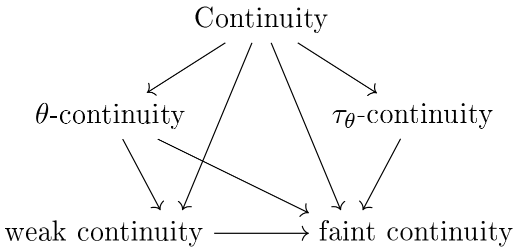
A function $f : X \to Y$ at the point $x \in X$ is

- **continuous** iff $\forall V \in \tau(f(x))\ \exists U \in \tau(x)\ \ f[U] \subseteq V$
- **weakly continuous** iff $\forall V \in \tau(f(x))\ \exists U \in \tau(x)\ \ f[U] \subseteq \mathrm{Cl}(V)$
- $\theta$-**continuous** iff $\forall V \in \tau(f(x))\ \exists U \in \tau(x)\ \ f[\mathrm{Cl}(U)] \subseteq \mathrm{Cl}(V)$
- $\tau_\theta$-**continuous** iff $\forall V \in \tau_\theta(f(x))\ \exists U \in \tau_\theta(x)\ \ f[U] \subseteq V$
- **faintly-continuous** iff $\forall V \in \tau_\theta(f(x))\ \exists U \in \tau(x)\ \ f[U] \subseteq V$

## Definition

A function $f : X \to Y$ at the point $x \in X$ is

- **continuous** iff $\forall V \in \tau(f(x))\ \exists U \in \tau(x)\ \ f[U] \subseteq V$
- **weakly continuous** iff $\forall V \in \tau(f(x))\ \exists U \in \tau(x)\ \ f[U] \subseteq \mathrm{Cl}(V)$
- $\theta$**-continuous** iff $\forall V \in \tau(f(x))\ \exists U \in \tau(x)\ \ f[\mathrm{Cl}(U)] \subseteq \mathrm{Cl}(V)$
- $\tau_\theta$**-continuous** iff $\forall V \in \tau_\theta(f(x))\ \exists U \in \tau_\theta(x)\ \ f[U] \subseteq V$
- **faintly-continuous** iff $\forall V \in \tau_\theta(f(x))\ \exists U \in \tau(x)\ \ f[U] \subseteq V$



# Problem

**Question 1**

If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is continuous, what are sufficient conditions for $f : \langle X, \tau^* \rangle \to \langle Y, \sigma^* \rangle$ to remain continuous?

## Problem

**Question 1**

If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is continuous, what are sufficient conditions for $f : \langle X, \tau^* \rangle \to \langle Y, \sigma^* \rangle$ to remain continuous?

**Question 2**

If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is XXXXXXXX-continuous, what can we conclude about function if we change topologies by ideals?

# Continuity

### Theorem

Let $\langle X, \tau_X, \mathcal{I}_X\rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y\rangle$ be ideal topological spaces. If $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is a continuous function and for all $I \in \mathcal{I}_Y$ we have $f^{-1}[I] \in \mathcal{I}_X$. Then there hold the following equivalent conditions:

a) $\forall A \subseteq X \;\; f[A^*] \subseteq (f[A])^*$;
b) $\forall B \subseteq Y \;\; (f^{-1}[B])^* \subseteq f^{-1}[B^*]$.

which implies the following three equivalent conditions:

c) $\forall A \subseteq X \;\; f[\mathrm{Cl}^*(A)] \subseteq \mathrm{Cl}^*(f[A])$;
d) $\forall B \subseteq Y \;\; \mathrm{Cl}^*((f^{-1}[B])) \subseteq f^{-1}[\mathrm{Cl}^*(B)]$;

e) $f : \langle X, \tau^*_X\rangle \to \langle Y, \tau^*_Y\rangle$ is a continuous function.

# $\theta$-continuity

$$\forall V \in \tau(f(x))\ \exists U \in \tau(x) \quad f[\mathrm{Cl}(U)] \subseteq \mathrm{Cl}(V)$$

## Theorem

Let $\langle X, \tau_X, \mathcal{I}_X \rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y \rangle$ be ideal topological spaces. If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is a $\theta$-continuous function and for all $I \in \mathcal{I}_Y$ we have $f^{-1}[I] \in \mathcal{I}_X$, then there hold the following equivalent conditions:

a) $\forall A \subseteq X \ \ f[\Gamma(A)] \subseteq \Gamma(f[A])$;
b) $\forall B \subseteq Y \ \ \Gamma(f^{-1}[B]) \subseteq f^{-1}[\Gamma(B)]$.

which implies the following two equivalent conditions:

c) $\forall A \subseteq X \ \ f[\mathrm{Cl}_\sigma(A)] \subseteq \mathrm{Cl}_\sigma(f[A])$;
d) $f : \langle X, \sigma_X \rangle \to \langle Y, \sigma_Y \rangle$ is a continuous function.

## $\theta$-continuity

$$\forall V \in \tau(f(x))\ \exists U \in \tau(x)\ \ f[\mathrm{Cl}(U)] \subseteq \mathrm{Cl}(V)$$

### Theorem

Let $\langle X, \tau_X, \mathcal{I}_X \rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y \rangle$ be ideal topological spaces. If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is a $\theta$-continuous function and for all $I \in \mathcal{I}_Y$ we have $f^{-1}[I] \in \mathcal{I}_X$, then there hold the following equivalent conditions:

a) $\forall A \subseteq X\ \ f[\Gamma(A)] \subseteq \Gamma(f[A])$;
b) $\forall B \subseteq Y\ \Gamma(f^{-1}[B]) \subseteq f^{-1}[\Gamma(B)]$.

which implies the following two equivalent conditions:

c) $\forall A \subseteq X\ \ f[\mathrm{Cl}_\sigma(A)] \subseteq \mathrm{Cl}_\sigma(f[A])$;
d) $f : \langle X, \sigma_X \rangle \to \langle Y, \sigma_Y \rangle$ is a continuous function.

### Corollary

If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is a $\theta$-continuous function then $f : \langle X, (\tau_\theta)_X \rangle \to \langle Y, (\tau_\theta)_Y \rangle$ is continuous.

# Weak continuity

$$\forall V \in \tau(f(x))\ \exists U \in \tau(x)\ \ f[U] \subseteq \mathrm{Cl}(V)$$

## Theorem

Let $\langle X, \tau_X, \mathcal{I}_X \rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y \rangle$ be ideal topological spaces. If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is a weakly continuous function and for all $I \in \mathcal{I}_Y$ we have $f^{-1}[I] \in \mathcal{I}_X$, then there hold the following equivalent conditions:

a) $\forall A \subseteq X\ \ f[A^*] \subseteq \Gamma(f[A])$;

b) $\forall B \subseteq Y\ \ (f^{-1}[B])^* \subseteq f^{-1}[\Gamma(B)]$.

which implies the following condition:

c) $f : \langle X, \tau_X^* \rangle \to \langle Y, \sigma_Y \rangle$ is a continuous function.

# Weak continuity

$$\forall V \in \tau(f(x))\ \exists U \in \tau(x)\ \ f[U] \subseteq \mathrm{Cl}(V)$$

## Theorem

Let $\langle X, \tau_X, \mathcal{I}_X\rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y\rangle$ be ideal topological spaces. If $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is a weakly continuous function and for all $I \in \mathcal{I}_Y$ we have $f^{-1}[I] \in \mathcal{I}_X$, then there hold the following equivalent conditions:

a) $\forall A \subseteq X\ \ f[A^*] \subseteq \Gamma(f[A])$;

b) $\forall B \subseteq Y\ \ (f^{-1}[B])^* \subseteq f^{-1}[\Gamma(B)]$.

which implies the following condition:

c) $f : \langle X, \tau_X^*\rangle \to \langle Y, \sigma_Y\rangle$ is a continuous function.

## Corollary (Long and Herrington 1982)

If $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is a weakly continuous function then $f : \langle X, \tau_X\rangle \to \langle Y, (\tau_\theta)_Y\rangle$ is continuous, which is equivalent to faint continuity of $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$.

# Weakly-continuous vs. $\tau_\theta$-continuous

## Theorem

If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is weakly continuous and not $\tau_\theta$-continuous, then both $X$ and $Y$ have infinite topologies.

## Weakly-continuous vs. $\tau_\theta$-continuous

### Theorem

If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is weakly continuous and not $\tau_\theta$-continuous, then both $X$ and $Y$ have infinite topologies. So, if $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is weakly continuous and if $X$ or $Y$ is has finite topology, then $f$ is $\tau_\theta$-continuous.

# Weakly-continuous vs. $\tau_\theta$-continuous

## Theorem

If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is weakly continuous and not $\tau_\theta$-continuous, then both $X$ and $Y$ have infinite topologies. So, if $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is weakly continuous and if $X$ or $Y$ is has finite topology, then $f$ is $\tau_\theta$-continuous.

## Example (Weakly-continuous, but not $\tau_\theta$-continuous)

Let $X = \{x_0, x_1\} \cup \omega$ and $Y = \{y_0, y_1\} \cup \omega \times \{0, 1\}$.

$$
\begin{aligned}
\mathcal{B}_X(x_0) &= \{\{x_0\} \cup \omega \setminus K : |K| < \aleph_0\} \\
\mathcal{B}_X(x_1) &= \{\{x_1\} \cup \omega \setminus K : |K| < \aleph_0\} \\
\mathcal{B}_X(n) &= \{n\}
\end{aligned}
$$

# Weakly-continuous vs. $\tau_\theta$-continuous

## Theorem

If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is weakly continuous and not $\tau_\theta$-continuous, then both $X$ and $Y$ have infinite topologies. So, if $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is weakly continuous and if $X$ or $Y$ is has finite topology, then $f$ is $\tau_\theta$-continuous.

## Example (Weakly-continuous, but not $\tau_\theta$-continuous)

Let $X = \{x_0, x_1\} \cup \omega$ and $Y = \{y_0, y_1\} \cup \omega \times \{0, 1\}$.

$$\begin{aligned}
\mathcal{B}_X(x_0) &= \{\{x_0\} \cup \omega \setminus K : |K| < \aleph_0\} \\
\mathcal{B}_X(x_1) &= \{\{x_1\} \cup \omega \setminus K : |K| < \aleph_0\} \\
\mathcal{B}_X(n) &= \{n\} \\
\mathcal{B}_Y(y_0) &= \{\{y_0\} \cup \{\langle k, 0\rangle : k \geq n\} : n \in \omega\}, \\
\mathcal{B}_Y(y_1) &= \{\{y_1\} \cup ((\omega \times \{1\}) \setminus K) \cup \{\langle n, 0\rangle\} : |K| < \aleph_0, n \in \omega\}, \\
\mathcal{B}_Y(\langle n, 0\rangle) &= \{\langle n, 0\rangle\}, \\
\mathcal{B}_Y(\langle n, 1\rangle) &= \{\{y_1\} \cup ((\omega \times \{1\}) \setminus K) \cup \{\langle n, 0\rangle, \langle n, 1\rangle\} : |K| < \aleph_0, n \in \omega\}.
\end{aligned}$$

# Weakly-continuous vs. $\tau_\theta$-continuous

## Theorem

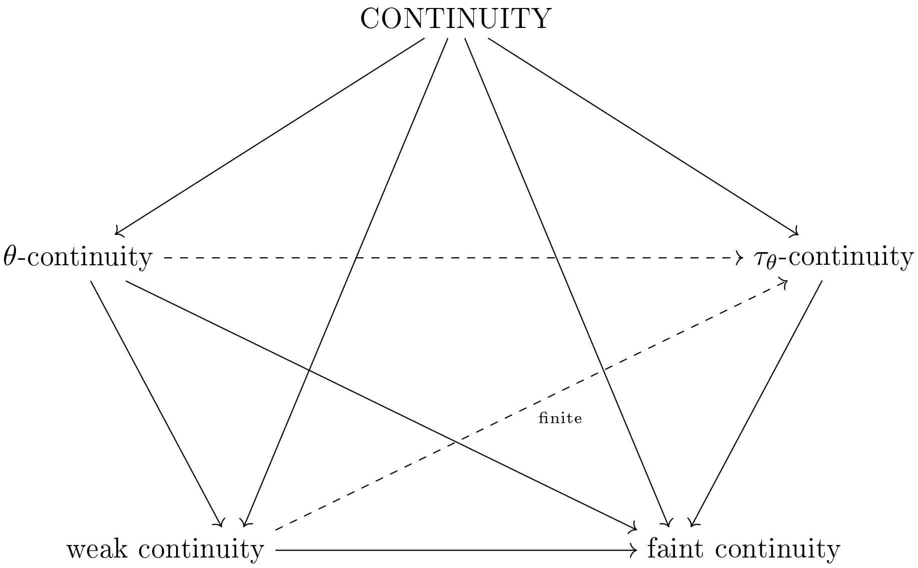
If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is weakly continuous and not $\tau_\theta$-continuous, then both $X$ and $Y$ have infinite topologies. So, if $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is weakly continuous and if $X$ or $Y$ is has finite topology, then $f$ is $\tau_\theta$-continuous.

## Example (Weakly-continuous, but not $\tau_\theta$-continuous)

Let $X = \{x_0, x_1\} \cup \omega$ and $Y = \{y_0, y_1\} \cup \omega \times \{0, 1\}$.

$$\begin{aligned}
\mathcal{B}_X(x_0) &= \{\{x_0\} \cup \omega \setminus K : |K| < \aleph_0\} \\
\mathcal{B}_X(x_1) &= \{\{x_1\} \cup \omega \setminus K : |K| < \aleph_0\} \\
\mathcal{B}_X(n) &= \{n\} \\
\mathcal{B}_Y(y_0) &= \{\{y_0\} \cup \{\langle k, 0\rangle : k \geq n\} : n \in \omega\}, \\
\mathcal{B}_Y(y_1) &= \{\{y_1\} \cup ((\omega \times \{1\}) \setminus K) \cup \{\langle n, 0\rangle\} : |K| < \aleph_0, n \in \omega\}, \\
\mathcal{B}_Y(\langle n, 0\rangle) &= \{\langle n, 0\rangle\}, \\
\mathcal{B}_Y(\langle n, 1\rangle) &= \{\{y_1\} \cup ((\omega \times \{1\}) \setminus K) \cup \{\langle n, 0\rangle, \langle n, 1\rangle\} : |K| < \aleph_0, n \in \omega\}.
\end{aligned}$$

$f(x_0) = y_0,\ f(x_1) = y_1,\ f(n) = \langle n, 1\rangle$, for $n \in \omega$.


CONTINUITY
θ-continuity
τθ-continuity
finite
weak continuity
faint continuity

Al-Omari, A., and Noiri, T.
Local closure functions in ideal topological spaces.
*Novi Sad J. Math. 43*, 2 (2013), 139–149.

Andrew, D. R., and Whittlesy, E. K.
Classroom Notes: Closure Continuity.
*Amer. Math. Monthly 73*, 7 (1966), 758–759.

Fomin, S.
Extensions of topological spaces.
*Ann. of Math. (2) 44* (1943), 471–480.

Janković, D., and Hamlett, T. R.
New topologies from old via ideals.
*Amer. Math. Monthly 97*, 4 (1990), 295–310.

Levine, N.
A decomposition of continuity in topological spaces.
*Amer. Math. Monthly 68* (1961), 44–46.

Long, P. E., and Herrington, L. L.
The $T_\theta$-topology and faintly continuous functions.
*Kyungpook Math. J. 22*, 1 (1982), 7–14.

Njamcul, A., and Pavlović, A.
On closure compatibility of ideal topological spaces and idempotency of the local closure function.
*Periodica Mathematica Hungarica 84*, 2 (2022), 221–234.

Njamcul, A., and Pavlović, A.
On preserving continuity in ideal topological spaces.
*Georgian Mathematical Journal 29*, 4 (2022), 567–574.

Njamcul, A., and Pavlović, A.
Various types of continuity and their interpretations in ideal topological spaces.
arXiv e-prints. doi:10.48550/arXiv.2212.01490 (2022)