Guessing models and forcing axioms

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Guessing models

Definition

Let $\theta \geq \omega_2$ be a regular cardinal and let $M \prec H(\theta)$ have size ω_1.

1. Given a set $x \in M$, and a subset $d \subseteq x$, we say that
   1. d is *M-approximated* if, for every $z \in M \cap \mathscr{P}_{\omega_1}(M)$, we have $d \cap z \in M$;
   2. d is *M-guessed* if there is $e \in M$ such that $d \cap M = e \cap M$.
2. M is a *guessing model for* x if every M-approximated subset of x is M-guessed.
3. M is a *guessing model* if M is guessing for every $x \in M$.

Guessing models: motivation

Intuitively, being a guessing model says that M is similar to the "outer universe" $H(\theta)$, if we restrict our attention to countable sets. For instance a subset d of ω_1 will be in M provided that all its countable initial segments are elements of M, or equivalently, $d \subseteq \omega_1$ will not be an element of M only if for some $\alpha < \omega_1$, $d \cap \alpha$ is not in M.

Note. Recall what it means that some (generic) extension of V satisfies the ω_1-approximation property. Using this concept, M is a guessing model iff the transitive collapse of M satisfies the ω_1-approximation property with respect to $H(\theta)$.

Guessing model property

Definition

We denote by GMP(θ) the assertion that the set

$$\{M \in \mathscr{P}_{\omega_2}(H(\theta)) \mid M \text{ is a guessing model}\}$$

is stationary in $\mathscr{P}_{\omega_2}(H(\theta))$. We write GMP if GMP($\theta$) holds for every regular $\theta \geq \omega_2$.

Guessing model property

- (Viale and Weiss) In the generic extension by Mitchell forcing up to a supercompact cardinal, GMP holds.
- In the generic extension by Mitchell forcing up to a weakly compact cardinal, GMP(ω_2) holds.
- (Viale and Weiss) PFA implies GMP.
- GMP implies $2^\omega > \omega_1$
- (Lambie-Hanson, S.) GMP implies $2^{\omega_1} = 2^\omega$ if $\mathrm{cf}(2^\omega) \neq \omega_1$, otherwise $2^{\omega_1} = (2^\omega)^+$.
- (Krueger) GMP implies SCH.
- (Lambie-Hanson, S.) GMP implies SSH.
- (Cox and Krueger) GMP(ω_3) implies $\neg$AP(ω_2) and TP(ω_2).

Kurepa and weak Kurepa trees

Recall the following definition:

Definition

Let κ be a cardinal. We say that a κ-tree T is a κ-*Kurepa tree* if it has at least κ^+-many cofinal branches; if we drop the restriction on T being a κ-tree, and require only that T has size and height κ, we obtain a *weak Kurepa tree*. We say that the *Kurepa Hypothesis*, KH(κ), holds if there exists a Kurepa tree on κ; analogously the *weak Kurepa Hypothesis*, wKH(κ), says that there exists a weak Kurepa tree on κ.

The negation of the weak Kurepa hypothesis

Some basic properties:

- If CH holds, then $2^{<\omega_1}$ is a weak Kurepa tree.
- Therefore $\neg$wKH(ω_1) implies $2^\omega > \omega_1$.
- (Mitchell) In the generic extension by Mitchell forcing up to an inaccessible cardinal $\neg$wKH(ω_1) holds.
- (Silver) The inaccessible cardinal is necessary. If $\neg$wKH(ω_1) holds, then ω_2 is inaccessible in L.

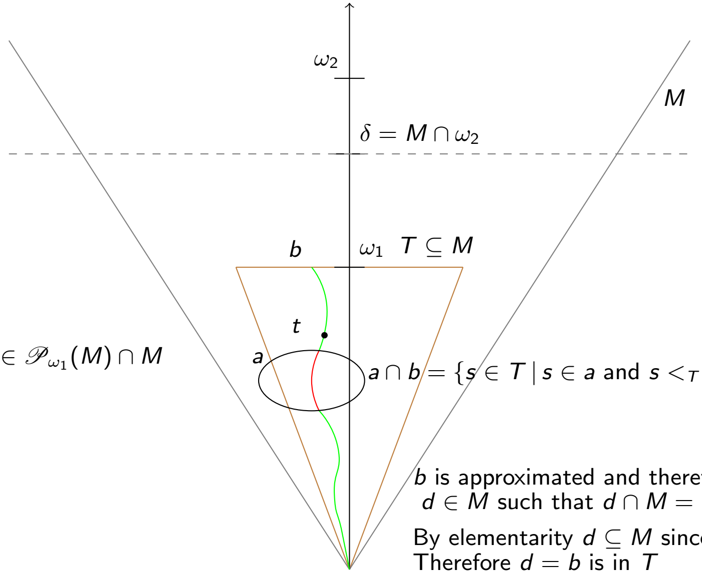
The negation of the weak Kurepa hypothesis

Assume $\neg\mathsf{wKH}(\omega_1)$ holds:

- (Baumgartner) If $2^\omega = \omega_2$, then $2^{\omega_1} = \omega_2$; in fact, even $\diamondsuit^+(\omega_2 \cap \mathrm{cof}(\omega_1))$ holds.
- Baumgartner's result can be generalized as follows: if $2^\omega < \aleph_{\omega_1}$, then $2^{\omega_1} = 2^\omega$.
- (Cox and Krueger) $\mathsf{GMP}(\omega_2)$ implies $\neg\mathsf{wKH}(\omega_1)$. We wil sketch a proof of the result of Cox and Krueger to illustrate the use of guessing models.

GMP(ω_2) implies $\neg$wKH(ω_1)

$M \prec H(\omega_2)$ is a guessing model such that $T \in M$, $|M| = \omega_1$ and $\omega_1 \subseteq M$



b is approximated and therefore there is $d \in M$ such that $d \cap M = b \cap M = b$.

By elementarity $d \subseteq M$ since $d \cap M \subseteq T$. Therefore $d = b$ is in T

Definition

Let $\theta \geq \omega_2$ be a regular cardinal. $M \in \mathscr{P}_{\omega_2} H(\theta)$ is said to be an *indestructible* ω_1*-guessing model* if it is an ω_1-guessing model and remains an ω_1-guessing model in any forcing extension that preserves ω_1. IGMP(θ) is the assertion that there are stationarily many indestructible guessing models in $\mathscr{P}_{\omega_2} H(\theta)$. IGMP is the assertion that IGMP(θ) holds for all regular $\theta \geq \omega_2$.

IGMP

- (Cox and Krueger) PFA implies IGMP; in particular IGMP follows from the conjunction of GMP and the assertion that all trees of height and size ω_1 are special.
- (Cox and Krueger) IGMP is compatible with any possible value of the continuum with cofinality at least ω_2.
- (Cox and Krueger) IGMP implies SH.

1. Cox and Krueger ask whether IGMP implies that the pseudointersection number $\mathfrak{p}$ is greater than ω_1.
2. Cox and Krueger ask whether IGMP implies that every tree of height and size ω_1 with no cofinal branches is special.
3. Krueger asks whether PFA(T^*) implies $\neg$wKH.

- To answer these question we work with forcing axioms for Suslin and almost Suslin trees.

Definition

Suppose that T is an ω_1-tree, i.e., a tree of height ω_1, all of whose levels are countable.

1. T is an *Aronszajn tree* if it has no cofinal branches.
2. T is a *Suslin tree* if it is an Aronszajn tree and has no uncountable antichains.
3. T is an *almost Suslin tree* if it has no stationary antichains, i.e., no antichains $A \subseteq T$ for which the set $\{\mathrm{ht}(s) \mid s \in A\}$ is stationary in ω_1, where $\mathrm{ht}(s)$ denotes the level of s in T.

Let S denote a Suslin tree and T^* an almost Suslin Aronszajn tree.

Definition

Let $\mathbb{P}$ be a forcing notion.

1. For a Suslin tree S, we say that $\mathbb{P}$ is *S-preserving* if $\Vdash_{\mathbb{P}}$ "S is a Suslin tree".
2. For an almost Suslin Aronszajn tree T^*, we say that $\mathbb{P}$ is *T^*-preserving* if $\Vdash_{\mathbb{P}}$ "T^* is an almost Suslin Aronszajn tree".

Forcing axioms

Definition

If $\mathcal{C}$ is a class of forcing posets, then FA($\mathcal{C}$) is the assertion that, for every $\mathbb{P} \in \mathcal{C}$ and every collection $\mathcal{D} = \{D_\alpha \mid \alpha < \omega_1\}$ of ω_1-many dense subsets of $\mathbb{P}$, there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

1. MA_{ω_1}(S) is the assertion that S is a Suslin tree and FA($\mathcal{C}$) holds, where $\mathcal{C}$ is the class of c.c.c. S-preserving posets.
2. PFA(S) is the assertion that S is a Suslin tree and FA($\mathcal{C}$) holds, where $\mathcal{C}$ is the class of proper S-preserving posets.
3. PFA(T*) is the assertion that T^* is an almost Suslin Aronszajn tree and FA($\mathcal{C}$) holds, where $\mathcal{C}$ is the class of proper T^*-preserving poset.

PFA(S)[S]

If we start with a model satisfying PFA(S) and then force with the Suslin tree S, then we say that the resulting forcing extension satisfies PFA(S)[S]. Asserting that PFA(S)[S] implies a statement φ should be understood as asserting that, in any model of ZFC satisfying PFA(S) for some Suslin tree S, we have $\Vdash_S \varphi$. MA_{ω_1}(S)[S] is defined analogously, with MA_{ω_1}(S) replacing PFA(S).

Forcing axioms for ω_1-trees

- PFA(S) implies $\mathfrak{p} > \omega_1$, PFA(S)[S] implies $\mathfrak{p} = \omega_1$.
- (Todorcevic) PFA(S) implies that there is a Suslin tree, PFA(S)[S] implies that all ω_1-trees are special.
- (Krueger) PFA(T^*) implies that there is a nonspecial ω_1-Aronszajn tree, but every ω_1-Aronszajn tree is special on cofinally many levels, in particular PFA(T^*) implies SH.

Our results

- (Lambie-Hanson, S.) PFA(S)[S] implies IGMP. This answers question (1) of Cox and Krueger negatively since, in any model of PFA(S)[S], we have $\mathfrak{p} = \omega_1$.
- (Lambie-Hanson, S.) PFA(S) implies GMP.
- (Lambie-Hanson, S.) PFA(T*) implies IGMP. This shows that IGMP does not imply that every tree of height and size ω_1 with no cofinal branches is special, which answers negatively question (2) of Cox and Krueger. In any model of PFA(T*), IGMP holds and T^* is a nonspecial Aronszajn tree.
- (Lambie-Hanson, S.) PFA(T*) implies $\neg$wKH, since IGMP implies $\neg$wKH. This answers question (3) of Krueger positively.

Open question

Cox and Krueger proved that IGMP is compatible with any possible value of the continuum with cofinality at least ω_2.

1. Is IGMP compatible with $\mathrm{cf}(2^\omega) = \omega_1$? What about just IGMP($\omega_2$)?

Motivated by this question we proved only that the "indestructible version" of $\neg$wKH is compatible with any possible value of the continuum, including values of cofinality ω_1.