On antiramsey colorings and geometry of Banach spaces

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based on a joint work with Piotr Koszmider[1]

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- A set theorist (or two)

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Suppose X is a Banach space ( ) with $\mathrm{dens}(X) \geq \kappa$. Is there an equilateral $A \subseteq X$ of size κ? Is there a $(1+\varepsilon)$-separated $A \subseteq S_X$[2] of size κ (for some $\varepsilon > 0$)?

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Suppose X is a Banach space (having some nice property) with $\mathrm{dens}(X) \geq \kappa$. Is there an equilateral $A \subseteq X$ of size κ? Is there a $(1 + \varepsilon)$-separated $A \subseteq S_X$[2] of size κ (for some $\varepsilon > 0$)?

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- Elton, Odell: The unit sphere of every infinite-dimensional Banach space contains an infinite $(1+\varepsilon)$-separated set (for some $\varepsilon > 0$ depending on the space).

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- Many examples of nonseparable Banach spaces without uncountable equilateral sets,
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The problem: how nice (how close to being reflexive) can nonseparable spaces without uncountable equilateral and $(1+\varepsilon)$-separated sets be?

Let $c\colon [\omega_1]^2 \to \{0,1\}$ be a coloring without uncountable monochromatic sets.[3] Put $\mathcal{A}_c = \{A \in [\omega_1]^{<\omega} : c[[A]^2] \subseteq \{0\}\}$.

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Let $\mathcal{X}_c$ be the completion of $(c_{00}(\omega_1), \|\cdot\|_c)$.

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A coloring $c\colon [\omega_1]^2 \to \{0, 1\}$ is a strong T-coloring if given any uncountable pairwise disjoint $\mathcal{F} \subseteq [\omega_1]^{<\omega}$ there are distinct $A, B \in \mathcal{F}$ such that $c[A \otimes B] = \{0\}$ and there are distinct $A', B' \in \mathcal{F}$ such that $c[A' \otimes B'] = \{1\}$.

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- Kojman, Rinot, Steprāns: If $\mathrm{non}(\mathcal{M}) = \omega_1$, then there is a strong T-coloring.
- Folklore?: Under $MA + \neg CH$ there is no strong T-coloring.

Lemma

Let c be a strong T-coloring and let $\{x_\alpha : \alpha < \omega_1\}$ be an uncountable sequence of vectors with finite, pairwise disjoint supports such that for some $r > 0$ and every $\alpha < \omega_1$ we have $\|x_\alpha\|_c = r$. Then there are $\alpha < \beta < \omega_1$ such that $\|x_\alpha - x_\beta\|_c = \sqrt{2}r$ and there are $\xi < \eta < \omega_1$ such that $\|x_\xi - x_\eta\|_c = r$.

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Proposition (P. Koszmider, KR)

For every $\delta > 0$ there is $\varepsilon > 0$ such that for every $(1-\varepsilon)$-separated $\{x_\alpha : \alpha < \omega_1\} \subseteq S_{\mathcal{X}_c}$ there are $\alpha < \beta < \omega_1$ such that $\|x_\alpha - x_\beta\|_c > \sqrt{2} - \delta$ and there are $\xi < \eta < \omega_1$ such that $\|x_\xi - x_\eta\|_c < 1 + \delta$.

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Let c be a strong T-coloring. Then the space $(\ell_2(\omega_1), \|\cdot\|_2 + \|\cdot\|_c)$ doesn't contain any uncountable equilateral sets.

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- is Hilbert generated and contains an isomorphic copy of ℓ_2 in every nonseparable subspace.

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- doesn't admit any uncountable $(1+\varepsilon)$-separated sets in its unit sphere.

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- doesn't admit any uncountable equilateral sets.
- doesn't admit any uncountable $(1+\varepsilon)$-separated sets in its unit sphere.

Theorem (P. Koszmider, KR)

Assume $MA + \neg CH$. Then for every coloring c the unit sphere of the space $\mathcal{X}_c$ contains an uncountable $\sqrt{2}$-equilateral set.

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Moreover, let $\mathcal{D}_c$ be the set of all finite families of consecutive pairs of countable ordinals $\{\xi_1, \eta_1\}, \ldots, \{\xi_k, \eta_k\}$ with $\xi_i < \eta_i$ for $1 \leq i \leq k$ and some $k \in \mathbb{N}$ such that for every $1 \leq i < j \leq k$ we have

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For $x \in c_{00}(\omega_1)$ put

$$\nu_c(x) = \sup_{D \in \mathcal{D}_c} \left(\sum_{\{\alpha,\beta\} \in D} |x(\alpha) - x(\beta)|^2 \right)^{1/2}.$$

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- There is a coloring c such that every bounded operator $T\colon \mathcal{Y}_c \to \mathcal{Y}_c$ is a scalar multiple of the identity plus a separable range operator and the unit sphere of $\mathcal{Y}_c$ contains an uncountable $\frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}+1}$-equilateral set.

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- Is there an equivalent renorming of $\ell_2(\omega_1)$ without uncountable equilateral sets?
- Is there an equivalent renorming of $c_0(\omega_1)$ without uncountable equilateral sets?

Thank you for your attention!