Countable discrete extensions of compact lines

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Twisted sums

Consider an exact sequence of Banach spaces

$$0 \to A \xrightarrow{j} X \xrightarrow{p} B \to 0,$$

consisting of Banach spaces and bounded linear operators for which the image of previous map is equal to the kernel of the next one. Such a sequence or the space X itself is called a *twisted sum* of spaces A and B.

Example

For any Banach spaces A, B there is a twisted sum $X = A \oplus B$.

Definition

The exact sequence is nontrivial if $j[A]$ is not complemented in X.

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CCKY Problem

Problem (Cabello Sánchez, Castillo, Kalton, Yost)

Is there a nontrivial twisted sum of c_0 and $C(K)$ for a nonmetrizable compact space K?

Problem

Given a cardinal number κ, is there a nontrivial twisted sum of c_0 and $C(K)$ for a nonmetrizable compact space K of weight equal to κ?

Fact

Assume that K is a compact space and there is $L \in CDE(K)$ without property $(\mathcal{E})$. Then there is a nontrivial twisted sum of c_0 and $C(K)$.

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Compact lines

Definition

Consider a closed subset $F \subseteq [0,1]$, any set $X \subseteq F$ and define a space

$$F_X = F \times \{0\} \cup X \times \{1\}$$

equipped with the topology generated by the lexicographic order.

Theorem (Ostaszewski, 1974)

The space L is a separable compact linearly ordered space if and only if L is homeomorphic to F_X for some closed set $F \subseteq [0,1]$ and a subset $X \subseteq F$.

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Properties of compact lines

Properties of the space F_X

- F_X is a compact Hausdorff space,
- $w(F_X) = |X|$,
- F_X is separable,
- F_X is 0-dimensional if X is dense in F.

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Countable discrete extensions

Definition

Given a compact space K, we say that L is a countable discrete extension of K if the following are satisfied

1. K is a subspace of L,
2. L is compact,
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Properties $(\mathcal{R})$ and $(\mathcal{E})$

Definition

For a compact space K and $L \in CDE(K)$ we say that

- L has property $(\mathcal{R})$ if there is a continuous retraction r from L onto K.
- L has property $(\mathcal{E})$ if there is an extension operator E from $C(K)$ to $C(L)$.

Here by an extension operator we mean a bounded linear operator $E : C(K) \to C(L)$ such that $Ef|K = f$ and for every $f \in C(K)$.

Observation

$(\mathcal{R}) \implies (\mathcal{E})$, as $Ef = f \circ r$ is an extension operator.

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Summary

Results

Fix $\omega < \kappa < \mathfrak{c}$. There are:

- A countable discrete extension of the space F_X without property $(\mathcal{R})$ for some set X of cardinality κ.
- A countable discrete extension of the space F_X without property $(\mathcal{E})$ for some set X of cardinality κ if $\kappa > non(\mathcal{M})$.

Open problem

Assume that the set X is meager. Is there a countable discrete extension of the space F_X without property $(\mathcal{E})$?

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Assume that the set X is meager. Is there a countable discrete extension of the space F_X without property $(\mathcal{E})$?

CDE via Stone spaces

- $Q = \mathbb{Q} \cap [0, 1]$,
- $Q = \{q_n : n \in \omega\}$,
- for $x \in [0, 1]$ put $A_x = \{n \in \omega : q_n < x\}$,
- let $\mathcal{A}$ be the Boolean Algebra generated by sets A_x and the family of finite subsets of Q.

Then $Ult(\mathcal{A}/fin) \simeq [0, 1]_{(0,1)}$ and $Ult(\mathcal{A}) \in CDE(Ult(\mathcal{A}/fin))$.
This extension has property $(\mathcal{R})$, as

$$r(x) = \begin{cases} x & \text{for } x \in Ult(\mathcal{A}/fin), \\ (q_n, 1) & \text{for } x = n \notin Ult(\mathcal{A}/fin). \end{cases}$$

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The norm of an extension operator

Theorem

Suppose that

- *K is a 0-dimensional compact line,*
- *$L \in CDE(K)$,*
- *there is an extension operator $E : C(K) \to C(L)$ satisfying $||E|| < 2$.*

Then L has property $(\mathcal{R})$.

Theorem

For any uncountable $\kappa \leq \mathfrak{c}$ there is a countable discrete extension of a compact line of weight κ with property $(\mathcal{E})$ such that the minimal norm of an extension operator is equal to 3.

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CDE without property ($\mathcal{R}$)

Construction (Marciszewski)

- $T = 2^{<\omega}$; $N \subseteq 2^\omega$ - sequences with infinitely many ones,
- for $x \in [0,1]$ put $S_x = \{x|_{n-1} {}^\frown 0 : x(n-1) = 1\}$,
- $A_x = \{t \in T : t \leq x\} \setminus S_x$,
- denote by $\mathcal{A}_X$ the Boolean algebra of subsets of T generated by $\{A_x : x \in X\} \cup fin(T)$ for some set $X \subseteq N$.

$S_{\frac{1}{2}}$, $A_{\frac{1}{2}}$

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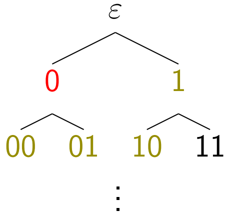

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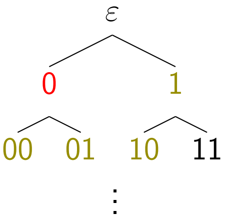

Properties of $Ult(\mathcal{A}_X)$

Theorem

The space $Ult(\mathcal{A}_X)$ is a CDE of a compact line without property $(\mathcal{R})$ for any uncountable set X.

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For any uncountable set X there is an extension operator $E : C(Ult(\mathcal{A}_X/fin)) \to C(Ult(\mathcal{A}_X))$ of the norm equal to 3.

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Property ($\mathcal{E}$)

Using a similar construction one can also prove the following theorem.

Theorem

For a second category set X there is a countable discrete extension of the space F_X without property ($\mathcal{E}$).