Random forcing, convergence of measures, and cofinality of Boolean algebras

Damian Sobota

Kurt Gödel Research Center for Mathematical Logic
University of Vienna

Joint work with Lyubomyr Zdomskyy.

Stone duality

Definition

A compact (Hausdorff) space K is *totally disconnected* (or *zero-dimensional*) if K has a basis of clopens.

Stone duality

Definition

A compact (Hausdorff) space K is *totally disconnected* (or *zero-dimensional*) if K has a basis of clopens.

Examples: 2^ω, $\beta\omega$, ω^*...

Stone duality

Definition

A compact (Hausdorff) space K is *totally disconnected* (or *zero-dimensional*) if K has a basis of clopens.

Examples: 2^ω, $\beta\omega$, ω^*...

Stone space

Let $\mathcal{A}$ be a Boolean algebra. The *Stone space* $St(\mathcal{A})$ of $\mathcal{A}$ is the space of all ultrafilters on $\mathcal{A}$ endowed with the topology generated by sets of the form:

$$[A]_\mathcal{A} = \{\mathcal{U} \in St(\mathcal{A}) \colon \ A \in \mathcal{U}\}$$

for every $A \in \mathcal{A}$.

Stone duality

Definition

A compact (Hausdorff) space K is *totally disconnected* (or *zero-dimensional*) if K has a basis of clopens.

Examples: 2^ω, $\beta\omega$, ω^*...

Stone space

Let $\mathcal{A}$ be a Boolean algebra. The *Stone space* $St(\mathcal{A})$ of $\mathcal{A}$ is the space of all ultrafilters on $\mathcal{A}$ endowed with the topology generated by sets of the form:

$$[A]_{\mathcal{A}} = \{\mathcal{U} \in St(\mathcal{A}) \colon \ A \in \mathcal{U}\}$$

for every $A \in \mathcal{A}$.

Facts

1. $St(\mathcal{A})$ is a totally disconnected compact space.

Stone duality

Definition

A compact (Hausdorff) space K is *totally disconnected* (or *zero-dimensional*) if K has a basis of clopens.

Examples: 2^ω, $\beta\omega$, ω^*...

Stone space

Let $\mathcal{A}$ be a Boolean algebra. The *Stone space* $St(\mathcal{A})$ of $\mathcal{A}$ is the space of all ultrafilters on $\mathcal{A}$ endowed with the topology generated by sets of the form:

$$[A]_{\mathcal{A}} = \{\mathcal{U} \in St(\mathcal{A}) \colon \ A \in \mathcal{U}\}$$

for every $A \in \mathcal{A}$.

Facts

1. $St(\mathcal{A})$ is a totally disconnected compact space.
2. $Clopen(St(\mathcal{A}))$ is isomorphic to $\mathcal{A}$.

σ-complete Boolean algebras

Definition

A compact space K is *basically disconnected* if every open $\mathbb{F}_\sigma$-set has open closure.

σ-complete Boolean algebras

Definition

A compact space K is *basically disconnected* if every open $\mathbb{F}_\sigma$-set has open closure.

Example: $\beta\omega$
Non-examples: 2^ω, ω^*

σ-complete Boolean algebras

Definition

A compact space K is *basically disconnected* if every open $\mathbb{F}_\sigma$-set has open closure.

Example: $\beta\omega$
Non-examples: 2^ω, ω^*

Definition

A Boolean algebra $\mathcal{A}$ is *σ-complete* if every countable subset of $\mathcal{A}$ has supremum in $\mathcal{A}$.

σ-complete Boolean algebras

Definition

A compact space K is *basically disconnected* if every open $\mathbb{F}_\sigma$-set has open closure.

Example: $\beta\omega$
Non-examples: 2^ω, ω^*

Definition

A Boolean algebra $\mathcal{A}$ is *σ-complete* if every countable subset of $\mathcal{A}$ has supremum in $\mathcal{A}$.

Examples: $\wp(\omega)$
Non-examples: $\mathit{Free}(\omega)$, $\wp(\omega)/\mathit{Fin}$

σ-complete Boolean algebras

Definition

A compact space K is *basically disconnected* if every open $\mathbb{F}_\sigma$-set has open closure.

Example: $\beta\omega$
Non-examples: 2^ω, ω^*

Definition

A Boolean algebra $\mathcal{A}$ is *σ-complete* if every countable subset of $\mathcal{A}$ has supremum in $\mathcal{A}$.

Examples: $\wp(\omega)$
Non-examples: $\mathit{Free}(\omega)$, $\wp(\omega)/\mathit{Fin}$

Fact

A Boolean algebra $\mathcal{A}$ is σ-complete if and only if $St(\mathcal{A})$ is basically disconnected.

Cohen reals and random forcing

Theorem (folklore — Dow and Fremlin point Koszmider)

Let $\mathcal{A} \in V$ be a ground model Boolean algebra. Let $\mathbb{P}$ be a notion of forcing adding a Cohen real.

Cohen reals and random forcing

Theorem (folklore — Dow and Fremlin point Koszmider)

Let $\mathcal{A} \in V$ be a ground model Boolean algebra. Let $\mathbb{P}$ be a notion of forcing adding a Cohen real. Then, in any $\mathbb{P}$-generic extension $V[G]$, the Stone space $St(\mathcal{A})$ contains a non-trivial convergent sequence.

Cohen reals and random forcing

Theorem (folklore — Dow and Fremlin point Koszmider)

Let $\mathcal{A} \in V$ be a ground model Boolean algebra. Let $\mathbb{P}$ be a notion of forcing adding a Cohen real. Then, in any $\mathbb{P}$-generic extension $V[G]$, the Stone space $St(\mathcal{A})$ contains a non-trivial convergent sequence.

Random forcing

$\kappa \geqslant \omega$ — a cardinal number

λ_κ — the standard product measure on 2^κ

$\mathbb{M}_\kappa = Bor(2^\kappa)/\{A \in Bor(2^\kappa) \colon \ \lambda_\kappa(A) = 0\}$

Cohen reals and random forcing

Theorem (folklore — Dow and Fremlin point Koszmider)

Let $\mathcal{A} \in V$ be a ground model Boolean algebra. Let $\mathbb{P}$ be a notion of forcing adding a Cohen real. Then, in any $\mathbb{P}$-generic extension $V[G]$, the Stone space $St(\mathcal{A})$ contains a non-trivial convergent sequence.

Random forcing

$\kappa \geqslant \omega$ — a cardinal number

λ_κ — the standard product measure on 2^κ

$\mathbb{M}_\kappa = Bor(2^\kappa)/\{A \in Bor(2^\kappa)\colon\ \lambda_\kappa(A) = 0\}$

Theorem (Dow–Fremlin)

Let $\mathcal{A} \in V$ be a ground model σ-complete Boolean algebra.

Cohen reals and random forcing

Theorem (folklore — Dow and Fremlin point Koszmider)

Let $\mathcal{A} \in V$ be a ground model Boolean algebra. Let $\mathbb{P}$ be a notion of forcing adding a Cohen real. Then, in any $\mathbb{P}$-generic extension $V[G]$, the Stone space $St(\mathcal{A})$ contains a non-trivial convergent sequence.

Random forcing

$\kappa \geqslant \omega$ — a cardinal number

λ_κ — the standard product measure on 2^κ

$\mathbb{M}_\kappa = Bor(2^\kappa)/\{A \in Bor(2^\kappa) \colon \ \lambda_\kappa(A) = 0\}$

Theorem (Dow–Fremlin)

Let $\mathcal{A} \in V$ be a ground model σ-complete Boolean algebra. Then, in any $\mathbb{M}_\kappa$-generic extension $V[G]$, the Stone space $St(\mathcal{A})$ does not contain any non-trivial convergent sequences.

Sequences of measures

K — compact space, $x \in K$, $A \subseteq K$

$$\delta_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Sequences of measures

K — compact space, $x \in K$, $A \subseteq K$

$$\delta_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Observation

Let K be a totally disconnected compact space, $\langle x_n \colon n \in \omega \rangle$ a non-trivial sequence in K, and $x \in K$. Then:

Sequences of measures

K — compact space, $x \in K$, $A \subseteq K$

$$\delta_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Observation

Let K be a totally disconnected compact space, $\langle x_n \colon\ n \in \omega \rangle$ a non-trivial sequence in K, and $x \in K$. Then:

$$x_n \to x \iff \forall \text{ clopen set } U \subseteq K\colon \delta_{x_n}(U) \to \delta_x(U)$$

Sequences of measures

K — compact space, $x \in K$, $A \subseteq K$

$$\delta_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Observation

Let K be a totally disconnected compact space, $\langle x_n \colon n \in \omega \rangle$ a non-trivial sequence in K, and $x \in K$. Then:

$$x_n \to x \iff \forall \text{ clopen set } U \subseteq K\colon \delta_{x_n}(U) \to \delta_x(U)$$

$$\implies \forall \text{ clopen set } U \subseteq K\colon \tfrac{1}{2}\delta_{x_{2n}}(U) - \tfrac{1}{2}\delta_{x_{2n+1}}(U) \to 0.$$

Sequences of measures

Definition

Let K be a totally disconnected compact space. A sequence $\langle \mu_n \colon n \in \omega \rangle$ of Borel measures on K such that:

- each $\mu_n = \sum_{x \in F_n} \alpha_x \delta_x$, where $F_n \in [K]^{<\omega}$ (*finite support*) and $\sum_{x \in F_n} |\alpha_x| = 1$,

Sequences of measures

Definition

Let K be a totally disconnected compact space. A sequence $\langle \mu_n \colon\ n \in \omega \rangle$ of Borel measures on K such that:

- each $\mu_n = \sum_{x \in F_n} \alpha_x \delta_x$, where $F_n \in [K]^{<\omega}$ (*finite support*) and $\sum_{x \in F_n} |\alpha_x| = 1$,
- $\mu_n(U) \to 0$ for every clopen set $U \subseteq K$,

Sequences of measures

Definition

Let K be a totally disconnected compact space. A sequence $\langle \mu_n\colon\ n \in \omega \rangle$ of Borel measures on K such that:

- each $\mu_n = \sum_{x \in F_n} \alpha_x \delta_x$, where $F_n \in [K]^{<\omega}$ (*finite support*) and $\sum_{x \in F_n} |\alpha_x| = 1$,
- $\mu_n(U) \to 0$ for every clopen set $U \subseteq K$,

is called *a finitely supported Josefson–Nissenzweig sequence* on K, or, in short, *a JN-sequence*.

Sequences of measures

Definition

Let K be a totally disconnected compact space. A sequence $\langle \mu_n \colon n \in \omega \rangle$ of Borel measures on K such that:

- each $\mu_n = \sum_{x \in F_n} \alpha_x \delta_x$, where $F_n \in [K]^{<\omega}$ (*finite support*) and $\sum_{x \in F_n} |\alpha_x| = 1$,
- $\mu_n(U) \to 0$ for every clopen set $U \subseteq K$,

is called *a finitely supported Josefson–Nissenzweig sequence* on K, or, in short, *a JN-sequence*.

Observation

Let $\langle \mu_n \colon n \in \omega \rangle$ be a bounded sequence of finitely supported measures on totally disconnected compact K. Then:

$$\forall \text{ clopen set } U \subseteq K \colon \mu_n(U) \to 0$$

Sequences of measures

Definition

Let K be a totally disconnected compact space. A sequence $\langle \mu_n \colon n \in \omega \rangle$ of Borel measures on K such that:

- each $\mu_n = \sum_{x \in F_n} \alpha_x \delta_x$, where $F_n \in [K]^{<\omega}$ (*finite support*) and $\sum_{x \in F_n} |\alpha_x| = 1$,
- $\mu_n(U) \to 0$ for every clopen set $U \subseteq K$,

is called *a finitely supported Josefson–Nissenzweig sequence* on K, or, in short, *a JN-sequence*.

Observation

Let $\langle \mu_n \colon n \in \omega \rangle$ be a bounded sequence of finitely supported measures on totally disconnected compact K. Then:

$$\forall \text{ clopen set } U \subseteq K \colon \mu_n(U) \to 0$$

$$\iff \forall f \in C(K) \colon \int_K f \, \mathrm{d}\mu_n \to 0 \text{ (\textit{weak* convergence})}.$$

Interlude

Theorem (Josefson–Nissenzweig)

For every infinite-dimensional Banach space X there exists a sequence $\langle x_n^*\colon\ n \in \omega\rangle$ of continuous functionals in the dual space X^* such that $\|x_n^*\| = 1$ for every $n \in \omega$ and $x_n^*(x) \to 0$ for every $x \in X$.

Existence of JN-sequences on infinite compact spaces

Examples

- spaces with non-trivial convergent sequences

Existence of JN-sequences on infinite compact spaces

Examples

- spaces with non-trivial convergent sequences
- metric spaces

Existence of JN-sequences on infinite compact spaces

Examples

- spaces with non-trivial convergent sequences
- metric spaces
- Eberlein, Corson, Valdivia spaces...

Existence of JN-sequences on infinite compact spaces

Examples

- spaces with non-trivial convergent sequences
- metric spaces
- Eberlein, Corson, Valdivia spaces...
- products $K \times L$ of arbitrary spaces, in particular $\beta\omega \times \beta\omega$

Existence of JN-sequences on infinite compact spaces

Examples

- spaces with non-trivial convergent sequences
- metric spaces
- Eberlein, Corson, Valdivia spaces...
- products $K \times L$ of arbitrary spaces, in particular $\beta\omega \times \beta\omega$
- limits $\varprojlim \langle K_\alpha, \pi_\alpha^\beta \colon \ \alpha \leqslant \beta < \delta \rangle$ based on simple extensions

Existence of JN-sequences on infinite compact spaces

Examples

- spaces with non-trivial convergent sequences
- metric spaces
- Eberlein, Corson, Valdivia spaces...
- products $K \times L$ of arbitrary spaces, in particular $\beta\omega \times \beta\omega$
- limits $\varprojlim \langle K_\alpha, \pi^\beta_\alpha \colon \ \alpha \leqslant \beta < \delta \rangle$ based on simple extensions

Non-examples

- basically disconnected spaces, in particular $\beta\omega$

Existence of JN-sequences on infinite compact spaces

Examples

- spaces with non-trivial convergent sequences
- metric spaces
- Eberlein, Corson, Valdivia spaces...
- products $K \times L$ of arbitrary spaces, in particular $\beta\omega \times \beta\omega$
- limits $\varprojlim \langle K_\alpha, \pi^\beta_\alpha \colon \ \alpha \leqslant \beta < \delta \rangle$ based on simple extensions

Non-examples

- basically disconnected spaces, in particular $\beta\omega$
- F-spaces, in particular ω^*

Existence of JN-sequences on infinite compact spaces

Examples

- spaces with non-trivial convergent sequences
- metric spaces
- Eberlein, Corson, Valdivia spaces...
- products $K \times L$ of arbitrary spaces, in particular $\beta\omega \times \beta\omega$
- limits $\varprojlim \langle K_\alpha, \pi_\alpha^\beta \colon \ \alpha \leqslant \beta < \delta \rangle$ based on simple extensions

Non-examples

- basically disconnected spaces, in particular $\beta\omega$
- F-spaces, in particular ω^*
- K for which $C(K)$ is a Grothendieck space

Cohen reals and random forcing, again

Theorem

Forcing notions adding Cohen reals add JN-sequences of the form $\frac{1}{2}\delta_{x_n} - \frac{1}{2}\delta_x$ to the Stone spaces of ground model Boolean algebras.

Cohen reals and random forcing, again

Theorem

Forcing notions adding Cohen reals add JN-sequences of the form $\frac{1}{2}\delta_{x_n} - \frac{1}{2}\delta_x$ to the Stone spaces of ground model Boolean algebras.

Theorem

$\mathbb{M}_\kappa$ does not add JN-sequences of the form $\frac{1}{2}\delta_{x_n} - \frac{1}{2}\delta_x$ to the Stone spaces of ground model σ-complete Boolean algebras.

Cohen reals and random forcing, again

Theorem

Forcing notions adding Cohen reals add JN-sequences of the form $\frac{1}{2}\delta_{x_n} - \frac{1}{2}\delta_x$ to the Stone spaces of ground model Boolean algebras.

Theorem

$\mathbb{M}_\kappa$ does not add JN-sequences of the form $\frac{1}{2}\delta_{x_n} - \frac{1}{2}\delta_x$ to the Stone spaces of ground model σ-complete Boolean algebras.

Main Question

Does $\mathbb{M}_\kappa$ add JN-sequences of *some other* form to the Stone spaces of ground model (σ-complete) Boolean algebras?

Positive answer

Theorem

Let $\mathcal{A} \in V$ be a ground model Boolean algebra. Let $\mathbb{P}$ be a forcing adding a random real. Then, in any $\mathbb{P}$-generic extension $V[G]$, there is a JN-sequence on the Stone space $St(\mathcal{A})$.

Positive answer

Theorem

Let $\mathcal{A} \in V$ be a ground model Boolean algebra. Let $\mathbb{P}$ be a forcing adding a random real. Then, in any $\mathbb{P}$-generic extension $V[G]$, there is a JN-sequence on the Stone space $St(\mathcal{A})$.

Sketch of the proof

$r \in 2^\omega$ — a random real over V

$\langle x_n \colon\ n \in \omega \rangle \in V$ — a non-trivial sequence of ultrafilters on $\mathcal{A}$

Positive answer

Theorem

Let $\mathcal{A} \in V$ be a ground model Boolean algebra. Let $\mathbb{P}$ be a forcing adding a random real. Then, in any $\mathbb{P}$-generic extension $V[G]$, there is a JN-sequence on the Stone space $St(\mathcal{A})$.

Sketch of the proof

$r \in 2^\omega$ — a random real over V

$\langle x_n \colon n \in \omega \rangle \in V$ — a non-trivial sequence of ultrafilters on $\mathcal{A}$

$$\mu_n = \frac{1}{2^n} \sum_{i=2^n+1}^{2^{n+1}} (-1)^{r(i)+1} \cdot \delta_{x_i}$$

Positive answer

Theorem

Let $\mathcal{A} \in V$ be a ground model Boolean algebra. Let $\mathbb{P}$ be a forcing adding a random real. Then, in any $\mathbb{P}$-generic extension $V[G]$, there is a JN-sequence on the Stone space $St(\mathcal{A})$.

Sketch of the proof

$r \in 2^\omega$ — a random real over V

$\langle x_n\colon\ n \in \omega\rangle \in V$ — a non-trivial sequence of ultrafilters on $\mathcal{A}$

$$\mu_n = \frac{1}{2^n} \sum_{i=2^n+1}^{2^{n+1}} (-1)^{r(i)+1} \cdot \delta_{x_i}$$

Weak Law of Large Numbers + Borel–Cantelli Lemma $\Longrightarrow$
$\langle \mu_n\colon\ n \in \omega\rangle$ is a JN-sequence on $St(\mathcal{A})$ in $V[G]$

Positive answer

Theorem

Let $\mathcal{A} \in V$ be a ground model Boolean algebra. Let $\mathbb{P}$ be a forcing adding a random real. Then, in any $\mathbb{P}$-generic extension $V[G]$, there is a JN-sequence on the Stone space $St(\mathcal{A})$.

Sketch of the proof

$r \in 2^\omega$ — a random real over V

$\langle x_n\colon\ n \in \omega\rangle \in V$ — a non-trivial sequence of ultrafilters on $\mathcal{A}$

$$\mu_n = \frac{1}{2^n} \sum_{i=2^n+1}^{2^{n+1}} (-1)^{r(i)+1} \cdot \delta_{x_i}$$

Weak Law of Large Numbers + Borel–Cantelli Lemma $\Longrightarrow$
$\langle \mu_n\colon\ n \in \omega\rangle$ is a JN-sequence on $St(\mathcal{A})$ in $V[G]$

Observe: $|\mathrm{supp}(\mu_n)| = 2^n$ **! So,** $\lim_{n\to\infty} |\mathrm{supp}(\mu_n)| = \infty$**.**

Negative answer

Theorem

Let $\mathcal{A} \in V$ be a ground model σ-complete Boolean algebra.

Negative answer

Theorem

Let $\mathcal{A} \in V$ be a ground model σ-complete Boolean algebra. Then, in any $\mathbb{M}_\kappa$-generic extension $V[G]$, the Stone space $St(\mathcal{A})$ does not admit any JN-sequence $\langle \mu_n \colon\ n \in \omega \rangle$ for which there exists $M \in \omega$ such that $|\mathrm{supp}(\mu_n)| \leqslant M$ for all $n \in \omega$.

Negative answer

Theorem

Let $\mathcal{A} \in V$ be a ground model σ-complete Boolean algebra. Then, in any $\mathbb{M}_\kappa$-generic extension $V[G]$, the Stone space $St(\mathcal{A})$ does not admit any JN-sequence $\langle \mu_n\colon\ n \in \omega \rangle$ for which there exists $M \in \omega$ such that $|\mathrm{supp}(\mu_n)| \leqslant M$ for all $n \in \omega$.

Proposition

For every totally disconnected compact space K, TFAE:

1. K admits a JN-sequence $\langle \mu_n\colon\ n \in \omega \rangle$ such that $|\mathrm{supp}(\mu_n)| \leqslant M$ for some $M \in \omega$ and all $n \in \omega$;

Negative answer

Theorem

Let $\mathcal{A} \in V$ be a ground model σ-complete Boolean algebra. Then, in any $\mathbb{M}_\kappa$-generic extension $V[G]$, the Stone space $St(\mathcal{A})$ does not admit any JN-sequence $\langle \mu_n \colon\ n \in \omega \rangle$ for which there exists $M \in \omega$ such that $|\text{supp}(\mu_n)| \leqslant M$ for all $n \in \omega$.

Proposition

For every totally disconnected compact space K, TFAE:

1. K admits a JN-sequence $\langle \mu_n \colon\ n \in \omega \rangle$ such that $|\text{supp}(\mu_n)| \leqslant M$ for some $M \in \omega$ and all $n \in \omega$;
2. K admits a JN-sequence $\langle \mu_n \colon\ n \in \omega \rangle$ such that $|\text{supp}(\mu_n)| = 2$ for all $n \in \omega$.

Main ingredients of the proof

Theorem (Dow–Fremlin)

Let $\mathcal{A} \in V$ be a ground model Boolean algebra. Assume that $\langle \dot{\mathcal{U}}_n\colon\ n \in \omega \rangle$ is a sequence of $\mathbb{M}_\kappa$-names for distinct ultrafilters on $\mathcal{A}$. Let G be a $\mathbb{M}_\kappa$-generic filter over V.

Main ingredients of the proof

Theorem (Dow–Fremlin)

Let $\mathcal{A} \in V$ be a ground model Boolean algebra. Assume that $\langle \dot{\mathcal{U}}_n \colon \ n \in \omega \rangle$ is a sequence of $\mathbb{M}_\kappa$-names for distinct ultrafilters on $\mathcal{A}$. Let G be a $\mathbb{M}_\kappa$-generic filter over V. Then, for every condition $p \in \mathbb{M}_\kappa$ there are a condition $q \leqslant p$ and a sequence $\langle A_n \colon \ n \in \omega \rangle \in V$ of pairwise disjoint elements of $\mathcal{A}$ such that $q \Vdash [A_n]_{\mathcal{A}} \cap \{\dot{\mathcal{U}}_k \colon k \in \omega\} \neq \emptyset$ for every $n \in \omega$.

Main ingredients of the proof

Theorem (Dow–Fremlin)

Let $\mathcal{A} \in V$ be a ground model Boolean algebra. Assume that $\langle \dot{\mathcal{U}}_n \colon n \in \omega \rangle$ is a sequence of $\mathbb{M}_\kappa$-names for distinct ultrafilters on $\mathcal{A}$. Let G be a $\mathbb{M}_\kappa$-generic filter over V. Then, for every condition $p \in \mathbb{M}_\kappa$ there are a condition $q \leqslant p$ and a sequence $\langle A_n \colon n \in \omega \rangle \in V$ of pairwise disjoint elements of $\mathcal{A}$ such that $q \Vdash [A_n]_{\mathcal{A}} \cap \{\dot{\mathcal{U}}_k \colon k \in \omega\} \neq \emptyset$ for every $n \in \omega$.

Theorem (Borodulin-Nadzieja–S.)

Let $\mathcal{A} \in V$ be a ground model Boolean algebra. Let $\dot{\mathcal{U}}$ and $\dot{\mathcal{V}}$ be $\mathbb{M}_\kappa$-names for ultrafilters on $\mathcal{A}$. Let $p \in \mathbb{M}_\kappa$ be a condition such that $p \Vdash \dot{\mathcal{U}} \neq \dot{\mathcal{V}}$.

Main ingredients of the proof

Theorem (Dow–Fremlin)

Let $\mathcal{A} \in V$ be a ground model Boolean algebra. Assume that $\langle \dot{\mathcal{U}}_n \colon\ n \in \omega \rangle$ is a sequence of $\mathbb{M}_\kappa$-names for distinct ultrafilters on $\mathcal{A}$. Let G be a $\mathbb{M}_\kappa$-generic filter over V. Then, for every condition $p \in \mathbb{M}_\kappa$ there are a condition $q \leqslant p$ and a sequence $\langle A_n \colon\ n \in \omega \rangle \in V$ of pairwise disjoint elements of $\mathcal{A}$ such that $q \Vdash [A_n]_{\mathcal{A}} \cap \{\dot{\mathcal{U}}_k \colon k \in \omega\} \neq \emptyset$ for every $n \in \omega$.

Theorem (Borodulin-Nadzieja–S.)

Let $\mathcal{A} \in V$ be a ground model Boolean algebra. Let $\dot{\mathcal{U}}$ and $\dot{\mathcal{V}}$ be $\mathbb{M}_\kappa$-names for ultrafilters on $\mathcal{A}$. Let $p \in \mathbb{M}_\kappa$ be a condition such that $p \Vdash \dot{\mathcal{U}} \neq \dot{\mathcal{V}}$. Then, for every $\varepsilon > 0$ there are a condition $q \leqslant p$ and an element $C \in \mathcal{A}$ such that $\lambda_\kappa(q) > \lambda_\kappa(p)/4 - \varepsilon$ and $q \Vdash C \in \dot{\mathcal{U}} \triangle \dot{\mathcal{V}}$.

What about other forcings?

Definition

A forcing $\mathbb{P} \in V$ has *the Laver property* if for every $\mathbb{P}$-generic filter G over V, every $f \in \omega^\omega \cap V$ and $g \in \omega^\omega \cap V[G]$ such that $g \leqslant^* f$, there exists $H \colon \omega \to [\omega]^{<\omega}$ such that $g(n) \in H(n)$ and $|H(n)| \leqslant n + 1$ for every $n \in \omega$.

What about other forcings?

Definition

A forcing $\mathbb{P} \in V$ has *the Laver property* if for every $\mathbb{P}$-generic filter G over V, every $f \in \omega^\omega \cap V$ and $g \in \omega^\omega \cap V[G]$ such that $g \leqslant^* f$, there exists $H \colon \omega \to [\omega]^{<\omega}$ such that $g(n) \in H(n)$ and $|H(n)| \leqslant n + 1$ for every $n \in \omega$.

Examples: Sacks, side-by-side Sacks, Laver, Mathias, Miller, Silver(-like)

What about other forcings?

Definition

A forcing $\mathbb{P} \in V$ has *the Laver property* if for every $\mathbb{P}$-generic filter G over V, every $f \in \omega^\omega \cap V$ and $g \in \omega^\omega \cap V[G]$ such that $g \leqslant^* f$, there exists $H\colon \omega \to [\omega]^{<\omega}$ such that $g(n) \in H(n)$ and $|H(n)| \leqslant n+1$ for every $n \in \omega$.

Examples: Sacks, side-by-side Sacks, Laver, Mathias, Miller, Silver(-like)

Definition

A forcing $\mathbb{P} \in V$ *preserves the ground model reals non-meager* if $\mathbb{R} \cap V$ is a non-meager subset of $\mathbb{R} \cap V[G]$ for any $\mathbb{P}$-generic filter G.

What about other forcings?

Definition

A forcing $\mathbb{P} \in V$ has *the Laver property* if for every $\mathbb{P}$-generic filter G over V, every $f \in \omega^\omega \cap V$ and $g \in \omega^\omega \cap V[G]$ such that $g \leqslant^* f$, there exists $H\colon \omega \to [\omega]^{<\omega}$ such that $g(n) \in H(n)$ and $|H(n)| \leqslant n+1$ for every $n \in \omega$.

Examples: Sacks, side-by-side Sacks, Laver, Mathias, Miller, Silver(-like)

Definition

A forcing $\mathbb{P} \in V$ *preserves the ground model reals non-meager* if $\mathbb{R} \cap V$ is a non-meager subset of $\mathbb{R} \cap V[G]$ for any $\mathbb{P}$-generic filter G.

Examples: Sacks, side-by-side Sacks, Miller, Silver(-like)

What about other forcings?

Theorem (S.–Zdomskyy)

Let $\mathcal{A} \in V$ be a ground model σ-complete Boolean algebra. Let $\mathbb{P} \in V$ be a notion of proper forcing having the Laver property and preserving the ground model reals non-meager.

What about other forcings?

Theorem (S.–Zdomskyy)

Let $\mathcal{A} \in V$ be a ground model σ-complete Boolean algebra. Let $\mathbb{P} \in V$ be a notion of proper forcing having the Laver property and preserving the ground model reals non-meager. Then, in any $\mathbb{P}$-generic extension $V[G]$, $St(\mathcal{A})$ does not admit any JN-sequences.

Why do we care? Complemented copies of c_0!

$$c_0 = \{x \in \mathbb{R}^\omega : \; x(n) \to 0\}$$

Two topologies on c_0

- norm $\|x\|_\infty = \sup_{n \in \omega} |x(n)|$, making c_0 a Banach space
- pointwise topology τ_p inherited from $\mathbb{R}^\omega$

Why do we care? Complemented copies of c_0!

$$c_0 = \{x \in \mathbb{R}^\omega : \ x(n) \to 0\}$$

Two topologies on c_0

- norm $\|x\|_\infty = \sup_{n\in\omega} |x(n)|$, making c_0 a Banach space
- pointwise topology τ_p inherited from $\mathbb{R}^\omega$

Theorem (Banakh–Kąkol–Śliwa)

For every totally disconnected compact space K TFAE:

1. K admits a JN-sequence sequence,

Why do we care? Complemented copies of c_0!

$$c_0 = \{x \in \mathbb{R}^\omega : \; x(n) \to 0\}$$

Two topologies on c_0

- norm $\|x\|_\infty = \sup_{n\in\omega} |x(n)|$, making c_0 a Banach space
- pointwise topology τ_p inherited from $\mathbb{R}^\omega$

Theorem (Banakh–Kąkol–Śliwa)

For every totally disconnected compact space K TFAE:

1. K admits a JN-sequence sequence,
2. $C_p(K)$ has a complemented copy of (c_0, τ_p).

Why do we care? Complemented copies of c_0!

$$c_0 = \{x \in \mathbb{R}^\omega : \; x(n) \to 0\}$$

Two topologies on c_0

- norm $\|x\|_\infty = \sup_{n\in\omega} |x(n)|$, making c_0 a Banach space
- pointwise topology τ_p inherited from $\mathbb{R}^\omega$

Theorem (Banakh–Kąkol–Śliwa)

For every totally disconnected compact space K TFAE:

1. K admits a JN-sequence sequence,
2. $C_p(K)$ has a complemented copy of (c_0, τ_p).

Corollary (by the Closed Graph Theorem)

If K as above admits a JN-sequence, then the Banach space $C(K)$ has a complemented copy of $(c_0, \|\cdot\|_\infty)$.

Complemented copies of c_0 in random extensions

Corollary

Let $\mathcal{A} = \wp(\omega) \cap V$. Then,

1. in V, $C(St(\mathcal{A}))$ does not have any complemented copies of $(c_0, \|\cdot\|_\infty)$ (Sobczyk's theorem);

Complemented copies of c_0 in random extensions

Corollary

Let $\mathcal{A} = \wp(\omega) \cap V$. Then,

1. in V, $C(St(\mathcal{A}))$ does not have any complemented copies of $(c_0, \|\cdot\|_\infty)$ (Sobczyk's theorem);
2. in $V^{\mathbb{M}_\kappa}$, $C(St(\mathcal{A}))$ has a complemented copy of $(c_0, \|\cdot\|_\infty)$.

Complemented copies of c_0 in random extensions

Corollary

Let $\mathcal{A} = \wp(\omega) \cap V$. Then,

1. in V, $C(St(\mathcal{A}))$ does not have any complemented copies of $(c_0, \|\cdot\|_\infty)$ (Sobczyk's theorem);
2. in $V^{\mathbb{M}_\kappa}$, $C(St(\mathcal{A}))$ has a complemented copy of $(c_0, \|\cdot\|_\infty)$.

Theorem

If $\mathbb{P}$ is a proper notion of forcing having the Laver property and preserving ground model reals non-meager, then, in $V^{\mathbb{P}}$, $C(St(\mathcal{A}))$ does not have any complemented copies of $(c_0, \|\cdot\|_\infty)$.

Why do we care? Cofinality of Boolean algebras!

Definition

The cofinality $\mathrm{cf}(\mathcal{A})$ of an infinite Boolean algebra $\mathcal{A}$ is the minimal cardinality κ of an increasing chain $\langle \mathcal{A}_\xi \colon \xi < \kappa \rangle$ of proper subalgebras of $\mathcal{A}$ such that $\mathcal{A} = \bigcup_{\xi<\kappa} \mathcal{A}_\xi$.

Why do we care? Cofinality of Boolean algebras!

Definition

The cofinality $\mathrm{cf}(\mathcal{A})$ of an infinite Boolean algebra $\mathcal{A}$ is the minimal cardinality κ of an increasing chain $\langle \mathcal{A}_\xi : \ \xi < \kappa \rangle$ of proper subalgebras of $\mathcal{A}$ such that $\mathcal{A} = \bigcup_{\xi<\kappa} \mathcal{A}_\xi$.

Theorem (Koppelberg)

For any infinite Boolean algebra $\mathcal{A}$, $\omega \leqslant \mathrm{cf}(\mathcal{A}) \leqslant \mathfrak{c}$.

Why do we care? Cofinality of Boolean algebras!

Definition

The cofinality $\mathrm{cf}(\mathcal{A})$ of an infinite Boolean algebra $\mathcal{A}$ is the minimal cardinality κ of an increasing chain $\langle \mathcal{A}_\xi \colon \ \xi < \kappa \rangle$ of proper subalgebras of $\mathcal{A}$ such that $\mathcal{A} = \bigcup_{\xi<\kappa} \mathcal{A}_\xi$.

Theorem (Koppelberg)

For any infinite Boolean algebra $\mathcal{A}$, $\omega \leqslant \mathrm{cf}(\mathcal{A}) \leqslant \mathfrak{c}$.

Examples

- $|\mathcal{A}| \leqslant \omega \implies \mathrm{cf}(\mathcal{A}) = \omega$,

Why do we care? Cofinality of Boolean algebras!

Definition

The cofinality $\mathrm{cf}(\mathcal{A})$ of an infinite Boolean algebra $\mathcal{A}$ is the minimal cardinality κ of an increasing chain $\langle \mathcal{A}_\xi\colon\ \xi < \kappa\rangle$ of proper subalgebras of $\mathcal{A}$ such that $\mathcal{A} = \bigcup_{\xi<\kappa} \mathcal{A}_\xi$.

Theorem (Koppelberg)

For any infinite Boolean algebra $\mathcal{A}$, $\omega \leqslant \mathrm{cf}(\mathcal{A}) \leqslant \mathfrak{c}$.

Examples

- $|\mathcal{A}| \leqslant \omega \implies \mathrm{cf}(\mathcal{A}) = \omega$,
- $\mathcal{A}$ — σ-complete $\implies \mathrm{cf}(\mathcal{A}) = \omega_1$ (Koppelberg)

Why do we care? Cofinality of Boolean algebras!

Definition

The cofinality $\mathrm{cf}(\mathcal{A})$ of an infinite Boolean algebra $\mathcal{A}$ is the minimal cardinality κ of an increasing chain $\langle \mathcal{A}_\xi : \ \xi < \kappa \rangle$ of proper subalgebras of $\mathcal{A}$ such that $\mathcal{A} = \bigcup_{\xi<\kappa} \mathcal{A}_\xi$.

Theorem (Koppelberg)

For any infinite Boolean algebra $\mathcal{A}$, $\omega \leqslant \mathrm{cf}(\mathcal{A}) \leqslant \mathfrak{c}$.

Examples

- $|\mathcal{A}| \leqslant \omega \implies \mathrm{cf}(\mathcal{A}) = \omega$,
- $\mathcal{A}$ — σ-complete $\implies \mathrm{cf}(\mathcal{A}) = \omega_1$ (Koppelberg)

Open question (Koppelberg)

Does there consistently exist a Boolean algebra $\mathcal{A}$ such that $\mathrm{cf}(\mathcal{A}) > \omega_1$?

Why do we care? Cofinality of Boolean algebras!

Proposition (folklore)

For any infinite Boolean algebra $\mathcal{A}$, TFAE:

1. $\mathrm{cf}(\mathcal{A}) = \omega$,

Why do we care? Cofinality of Boolean algebras!

Proposition (folklore)

For any infinite Boolean algebra $\mathcal{A}$, TFAE:

1. $\mathrm{cf}(\mathcal{A}) = \omega$,
2. there is a non-trivial discrete sequence $\langle x_n \colon\ n \in \omega\rangle$ in $St(\mathcal{A})$ such that for all clopen $U \subseteq St(\mathcal{A})$ and almost all $n \in \omega$ we have:

$$x_{2n} \in U \iff x_{2n+1} \in U.$$

Why do we care? Cofinality of Boolean algebras!

Proposition (folklore)

For any infinite Boolean algebra $\mathcal{A}$, TFAE:

1. $\mathrm{cf}(\mathcal{A}) = \omega$,
2. there is a non-trivial discrete sequence $\langle x_n \colon\ n \in \omega \rangle$ in $St(\mathcal{A})$ such that for all clopen $U \subseteq St(\mathcal{A})$ and almost all $n \in \omega$ we have:

$$x_{2n} \in U \iff x_{2n+1} \in U.$$

Corollary

Let $\mathcal{A} \in V$ be a ground model σ-complete Boolean algebra. Then, in any $\mathbb{M}_\kappa$-generic extension $V[G]$, we have $\mathrm{cf}(\mathcal{A}) = \omega_1$.

The end

Thank you for the attention!