The Axiom of Choice and maximal δ-separated sets

Michał Dybowski

Warsaw University of Technology
Faculty of Mathematics and Information Sciences

Definition

Let $\delta > 0$. We say that a subset Y of a pseudometric space (X, d) is δ-`separated` set if $d(x, y) > \delta$ for all distinct points $x, y \in Y$.

Definition

Let $\delta > 0$. We say that a subset Y of a pseudometric space (X, d) is δ-`separated` set if $d(x, y) > \delta$ for all distinct points $x, y \in Y$.

It is easy to see that for every $\delta > 0$ an existence of a maximal (under inclusion "$\subset$") δ-separated set is guaranteed by Zorn's Lemma (so by the Axiom of Choice equivalently).

In fact, the following fact holds:

In fact, the following fact holds:

Proposition

The following statement is equivalent with **AC**:

(i) For every metric space (X, d) and $\delta > 0$ there exists a maximal δ-separated set.

In fact, the following fact holds:

Proposition

The following statement is equivalent with **AC**:

(i) For every metric space (X, d) and $\delta > 0$ there exists a maximal δ-separated set.

Proof.

(i) $\implies$ (**AC**):

In fact, the following fact holds:

Proposition

The following statement is equivalent with **AC**:

(i) For every metric space (X, d) and $\delta > 0$ there exists a maximal δ-separated set.

Proof.

(i) $\implies$ (**AC**): Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be an arbitrary family of nonempty pairwise disjoint sets and let $X = \bigcup_{\alpha \in \Lambda} A_\alpha$.

In fact, the following fact holds:

Proposition

The following statement is equivalent with **AC**:

(i) For every metric space (X, d) and $\delta > 0$ there exists a maximal δ-separated set.

Proof.

(i) $\implies$ (**AC**): Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be an arbitrary family of nonempty pairwise disjoint sets and let $X = \bigcup_{\alpha\in\Lambda} A_\alpha$. We define metric d on X as follows:

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 1/2, & \text{if } x \neq y \text{ and } x, y \in A_\alpha \text{ for some } \alpha \in \Lambda; \\ 1, & \text{otherwise.} \end{cases}$$

In fact, the following fact holds:

Proposition

The following statement is equivalent with **AC**:

(i) For every metric space (X, d) and $\delta > 0$ there exists a maximal δ-separated set.

Proof.

(i) $\implies$ (**AC**): Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be an arbitrary family of nonempty pairwise disjoint sets and let $X = \bigcup_{\alpha\in\Lambda} A_\alpha$. We define metric d on X as follows:

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 1/2, & \text{if } x \neq y \text{ and } x, y \in A_\alpha \text{ for some } \alpha \in \Lambda; \\ 1, & \text{otherwise.} \end{cases}$$

Then a maximal $3/4$-separated set in (X, d) contains exactly one element from each set A_α. □

In view of this proposition, it is natural to ask what is the set-theoretic status of an existence of maximal δ-separated sets in some classes of metric spaces.

In view of this proposition, it is natural to ask what is the set-theoretic status of an existence of maximal δ-separated sets in some classes of metric spaces.

Fact (ZF)

Let $\delta > 0$ and (X, d) be a pseudometric space such that all δ-separated sets in X are finite and their cardinalities are uniformly upper bounded by some constant C. Then, there exists a maximal δ-separated set.

In view of this proposition, it is natural to ask what is the set-theoretic status of an existence of maximal δ-separated sets in some classes of metric spaces.

Fact (**ZF**)

Let $\delta > 0$ and (X, d) be a pseudometric space such that all δ-separated sets in X are finite and their cardinalities are uniformly upper bounded by some constant C. Then, there exists a maximal δ-separated set.

Theorem (D., Górka)

The following statement is equivalent with **DC**:

($\star$) Let $\delta > 0$ and (X, d) be a (pseudo)metric space such that all δ-separated sets in X are finite. Then, there exists a maximal δ-separated set.

Fact (ZF)

Let $\delta > 0$ and (X, d) be a pseudometric space which contains a finite $\delta/2$-cover. Then, there exists a maximal δ-separated set.

Fact (ZF)

Let $\delta > 0$ and (X, d) be a pseudometric space which contains a finite $\delta/2$-cover. Then, there exists a maximal δ-separated set.

Theorem (D., Górka)

The following statement is equivalent with **DC**:

($\star$) Let $\delta > 0$ and (X, d) be a (pseudo)metric space which contains a countable $\delta/2$-cover. Then, there exists a maximal δ-separated set.

Fact (ZF)

Let $\delta > 0$ and (X, d) be a pseudometric space which contains a finite $\delta/2$-cover. Then, there exists a maximal δ-separated set.

Theorem (D., Górka)

The following statement is equivalent with **DC**:

($\star$) Let $\delta > 0$ and (X, d) be a (pseudo)metric space which contains a countable $\delta/2$-cover. Then, there exists a maximal δ-separated set.

Corollary (DC)

For every separable pseudometric space (X, d) and $\delta > 0$ there exists a maximal δ-separated set.

Fact (ZF)

Let $\delta > 0$ and (X, d) be a pseudometric space which contains a finite $\delta/2$-cover. Then, there exists a maximal δ-separated set.

Theorem (D., Górka)

The following statement is equivalent with **DC**:

($\star$) Let $\delta > 0$ and (X, d) be a (pseudo)metric space which contains a countable $\delta/2$-cover. Then, there exists a maximal δ-separated set.

Corollary (DC)

For every separable pseudometric space (X, d) and $\delta > 0$ there exists a maximal δ-separated set.

Problem

Is this corollary equivalent with **DC**?

Definition

We say that metric space X is `geometrically doubling` if there exists a constant $M \in \mathbb{N}$ such that for every $r > 0$ every ball of radius r can be covered by at most M balls of radius $r/2$.

Definition

We say that metric space X is `geometrically doubling` if there exists a constant $M \in \mathbb{N}$ such that for every $r > 0$ every ball of radius r can be covered by at most M balls of radius $r/2$.

Definition

Let (X, d) be a metric space. We say that the Borel measure μ on X is `doubling` if the measure of every open ball is finite and positive and there exists a constant $C \geq 1$ such that for every $x \in X$ and $r > 0$

$$\mu\left(B(x, 2r)\right) \leq C\mu\left(B(x, r)\right).$$

Some of the most famous theorems about these spaces:

Some of the most famous theorems about these spaces:

- Every geometrically doubling space is separable;

Some of the most famous theorems about these spaces:

- Every geometrically doubling space is separable;
- Every metric space which admits a doubling measure is geometrically doubling (Coifmann, Weiss);

Some of the most famous theorems about these spaces:

- Every geometrically doubling space is separable;
- Every metric space which admits a doubling measure is geometrically doubling (Coifmann, Weiss);
- Every compact geometrically doubling metric space carries a doubling measure (Volberg, Konyagin);

Some of the most famous theorems about these spaces:

- Every geometrically doubling space is separable;
- Every metric space which admits a doubling measure is geometrically doubling (Coifmann, Weiss);
- Every compact geometrically doubling metric space carries a doubling measure (Volberg, Konyagin);
- Every complete geometrically doubling metric space carries a doubling measure (Luukkainen, Saksman);

Some of the most famous theorems about these spaces:

- Every geometrically doubling space is separable;
- Every metric space which admits a doubling measure is geometrically doubling (Coifmann, Weiss);
- Every compact geometrically doubling metric space carries a doubling measure (Volberg, Konyagin);
- Every complete geometrically doubling metric space carries a doubling measure (Luukkainen, Saksman);
- For every geometrically doubling space (X, d) and $\varepsilon \in (0, 1)$ the space (X, d^{ε}) admits a bilipschitz embedding into $\mathbb{R}^N$ for some $N \in \mathbb{N}$ (Assouad).

Theorem (D., Górka)

The following statements are equivalent with **CC**:

Theorem (D., Górka)

The following statements are equivalent with **CC**:

(i) For every $\delta > 0$ and pseudometric space which admits a doubling measure there exists a maximal δ-separated set;

Theorem (D., Górka)

The following statements are equivalent with **CC**:

(i) For every $\delta > 0$ and pseudometric space which admits a doubling measure there exists a maximal δ-separated set;

(ii) For every $\delta > 0$ and geometrically doubling pseudometric space there exists a maximal δ-separated set;

Theorem (D., Górka)

The following statements are equivalent with **CC**:

(i) For every $\delta > 0$ and pseudometric space which admits a doubling measure there exists a maximal δ-separated set;

(ii) For every $\delta > 0$ and geometrically doubling pseudometric space there exists a maximal δ-separated set;

(iii) Every geometrically doubling pseudometric space is separable.

Theorem ($\lozenge$)

Let (X, d) be a pseudometric space. Then, the space X is separable if and only if there exists a Borel measure μ on X such that the measure of every open ball is positive and finite.

Theorem ($\Diamond$)

Let (X, d) be a pseudometric space. Then, the space X is separable if and only if there exists a Borel measure μ on X such that the measure of every open ball is positive and finite.

Proof of the impliaction $\implies$.

Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of X.

Theorem ($\diamondsuit$)

Let (X, d) be a pseudometric space. Then, the space X is separable if and only if there exists a Borel measure μ on X such that the measure of every open ball is positive and finite.

Proof of the impliaction $\implies$.

Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of X. Then we define Borel measure μ as follows:

$$\mu = \sum_{i=1}^{\infty} \frac{1}{2^i} \delta_{x_i}.$$

□

The known proofs of the reverse implication are based on the maximal δ-separated sets or Vitali $5r$-covering lemma which, in the general case, apply the Axiom of Choice.

The known proofs of the reverse implication are based on the maximal δ-separated sets or Vitali $5r$-covering lemma which, in the general case, apply the Axiom of Choice.

Theorem (D, Górka)

The implication $\Longleftarrow$ in the Theorem $\Diamond$ is equivalent with **CC**.