Compact connected spaces via the projective Fraïssé limit constructions

Aleksandra Kwiatkowska

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Lelek fan

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2. continuum - compact and connected metric space
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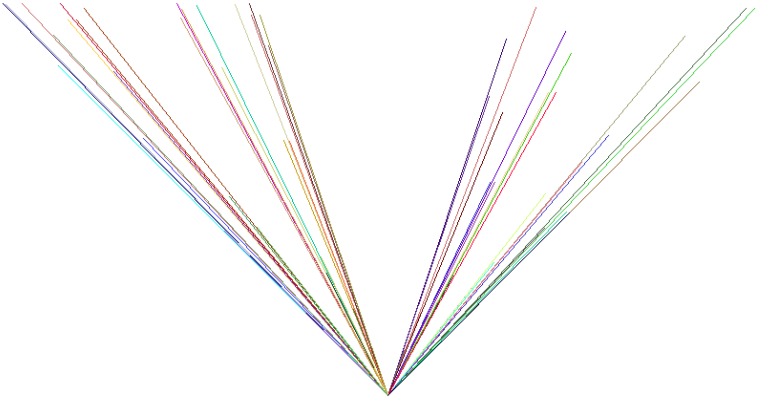
Lelek fan



Lelek fan from a projective Fraïssé limit, part 1

Let R be a binary relation symbol. Let $\mathcal{F}$ be the family of all finite reflexive fans.

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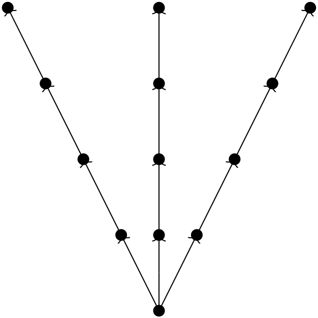

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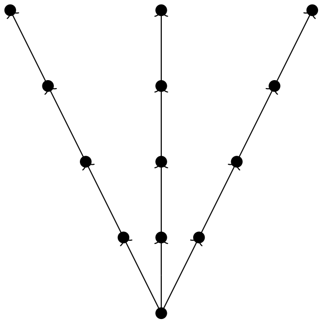

Theorem (Bartošová-K. '15)

$\mathcal{F}$ is a projective Fraïssé class.

Lelek fan from a projective Fraïssé limit, part 2

Lemma

Let $\mathbb{L}$ be the projective Fraïssé limit of $\mathcal{F}$. Then $R_S^{\mathbb{L}}$, where $R_S^{\mathbb{L}}(x, y)$ iff $R^{\mathbb{L}}(x, y)$ or $R^{\mathbb{L}}(y, x)$, is an equivalence relation such that each equivalence class has at most two elements.

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Theorem (Bartošová-K. '15)

$\mathbb{L}/R^{\mathbb{L}}$ is the Lelek fan.

Projective universality and Projective Ultrahomogeneity

smooth fan = subcontinuum of the Cantor fan that contains the top point

Theorem (Bartošová-K. '15)

1. *Each smooth fan is a continuous image of the Lelek fan.*
2. *Let X be a smooth fan. Let d be a metric on X. If f_1 and f_2 are increasing continuous surjections from the Lelek fan onto X, then for any $\epsilon > 0$ there exists a homeomorphism h of the Lelek fan such that for all x, $d(f_1(x), f_2 \circ h(x)) < \epsilon$.*

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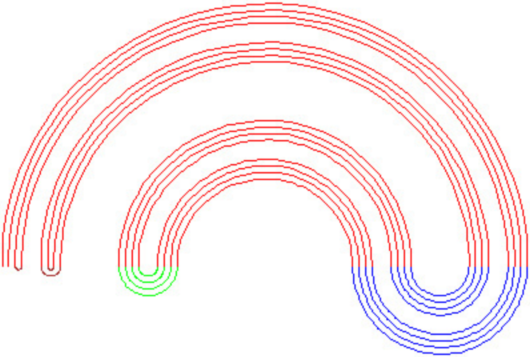
Remark

A somewhat related construction, of a compact space called Fraïssé fence, was given by Basso-Camerlo in 2021.

The buckethandle Knaster continuum



Knaster continua

Definition

A Knaster continuum is a continuum homeomorphic to the inverse limit $\varprojlim(I_n, f_n)$ of a sequence of unit intervals $I_n = [0,1]$ with continuous, open, non-homeomorphic surjections f_n that map 0 to 0.

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- Universal Knaster continuum is the Knaster continuum which continuously and openly surjects onto all Knaster continua.
- S. Iyer (2022) constructed the universal Knaster continuum as the topological realization of a projective Fraïssé limit.
- Another construction of the universal Knaster continuum in the projective Fraïssé theoretic framework was presented by L. Wickman (2022).

Topological graphs

Definition

A topological graph K is a graph $(V(K), E(K))$, whose domain $V(K)$ is a 0-dimensional, compact, second-countable (thus has a metric) space and $E(K)$ is a closed, reflexive and symmetric subset of $V(K)^2$.

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Definition

1. A continuous function $f : L \to K$ is a homomorphism if $\langle a, b\rangle \in E(L)$ implies $\langle f(a), f(b)\rangle \in E(K)$.
2. A homomorphism f is an epimorphism if it is moreover surjective on both vertices and edges.

Monotone maps

Definition

A subset S of a topological graph G is disconnected if there are two nonempty closed subsets P and Q of S such that $P \cup Q = S$ and if $a \in P$ and $b \in Q$, then $\langle a, b \rangle \notin E(G)$. A subset S of G is connected if it is not disconnected.

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Definition

- (continua) Let K, L be continua. A continuous map $f : L \to K$ is called monotone if for every subcontinuum M of K, $f^{-1}(M)$ is connected.
- (graphs) Let G, H be topological graphs. An epimorphism $f : G \to H$ is called monotone if for every closed connected subset Q of H, $f^{-1}(Q)$ is connected.

Menger sponge = universal Menger curve

Theorem (Menger '26)

The universal Menger curve is universal in the class of all metric separable spaces of dimension ≤ 1.

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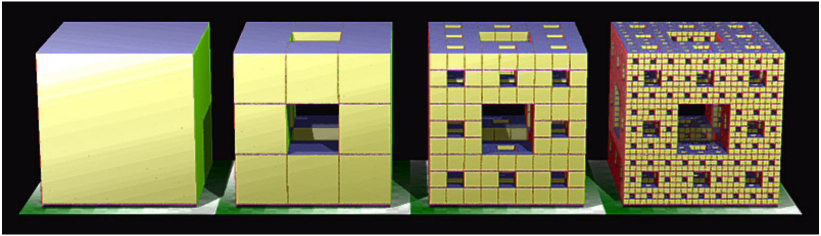
Theorem (Anderson '58)

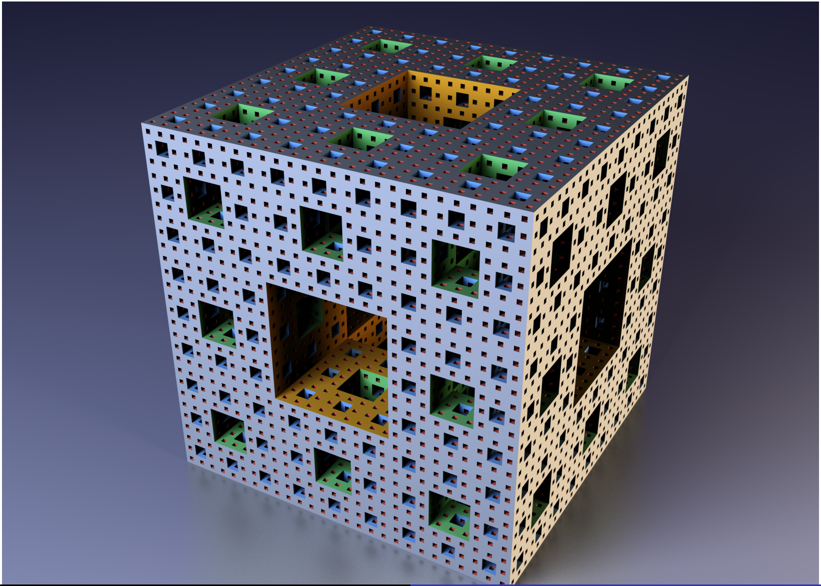
The following are equivalent for a continuum X.

1. X *is homeomorphic to the universal Menger curve,*
2. X *is a locally connected curve with no local cut points and no planar open nonempty subsets,*
3. X *is a homogeneous locally connected curve, which is not homeomorphic to a circle.*

Universal Menger curve - construction



Universal Menger curve - construction 2



Universal Menger curve - Fraïssé construction

Theorem (Panagiotopoulos-Solecki)

The class $\mathcal{F}$ of all finite connected graphs with monotone epimorphisms is a Fraïssé class. The topological realization of the projective Fraïssé limit of $\mathcal{F}$ is the universal Menger curve.

Let $\mathbb{M}$ denote the projective Fraïssé limit of $\mathcal{F}$.

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Theorem (Panagiotopoulos-Solecki '22)

1. *Each Peano continuum is a continuous monotone image of the universal Menger curve.*
2. *Let X be a Peano continuum. Let d be a metric on X. If f_1 and f_2 are continuous monotone surjections from the universal Menger curve onto X, then for any $\epsilon > 0$ there exists a homeomorphism h of the universal Menger curve such that for all x, $d(f_1(x), f_2 \circ h(x)) < \epsilon$.*

Homogeneity of the universal Menger curve

Definition

A topological subgraph K of $\mathbb{M}$ is locally non-separating if for each clopen connected W, the set $W \setminus K$ is connected.

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If K and L are saturated and locally non-separating subgraphs of $\mathbb{M}$, then each isomorphism from K to L extends to an automorphism of $\mathbb{M}$.

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Corollary (Anderson '58)

Any bijection between finite subsets of the universal Menger curve extends to a homeomorphism.

Proof.

Sketch of the proof on a blackboard. □