Compact connected spaces via the projective Fraïssé limit constructions

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Chainable continua 1

Definition

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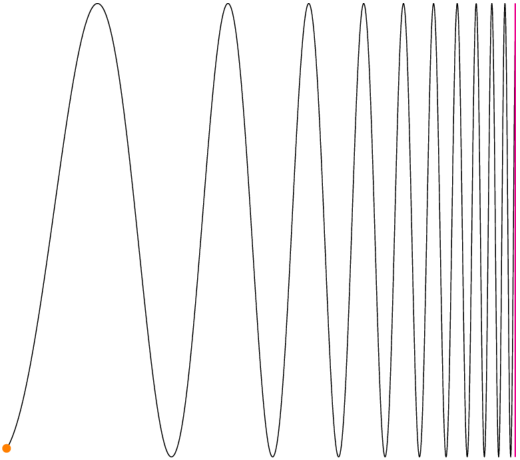
An open cover $\mathcal{U}$ *refines* an open cover $\mathcal{V}$ if every set from $\mathcal{U}$ is contained in some set in $\mathcal{V}$.

Definition

A continuum is chainable if any open cover can be refined by an open cover $U_1, \ldots, U_n$ such that for all $i, j \leq n$, we have $U_i \cap U_j \neq \emptyset$ iff $|i - j| \leq 1$.

$sin(\frac{1}{x})$-continuum is chainable



Chainable continua 2

Definition

A continuum X is arc-like if for every ϵ, there is a continuous and surjective $f\colon X \to [0,1]$ such that $f^{-1}(t)$ has diameter $< \epsilon$ for every $t \in [0,1]$.

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Theorem

A continuum is chainable iff it is arc-like iff it is an inverse limit of arcs [0,1] with continuous surjective bonding maps.

Indecomposable continua

Definition

A continuum is indecomposable if it is not the union of two proper subcontinua.

Definition

It is hereditarily indecomposable if its every subcontinuum is indecomposable.

Examples of indecomposable continua

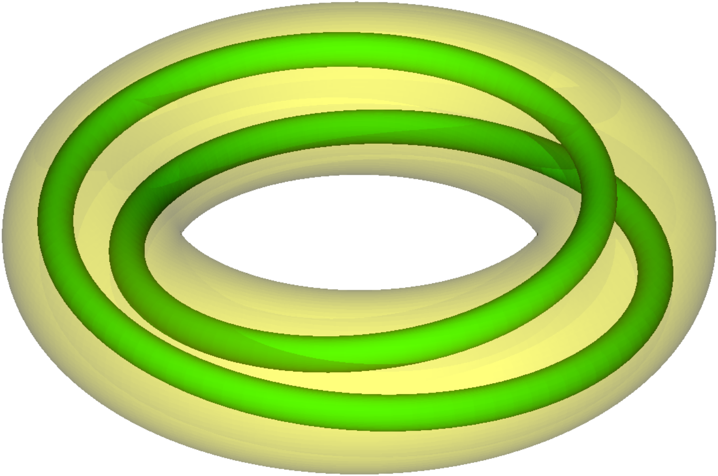
Proposition

Let $X = \varprojlim\{X_i, f_i\}$ be the inverse limit of continua. Suppose that for each i, whenever A_{i+1} and B_{i+1} are subcontinua of X_{i+1} and all subcontina such that $X_{i+1} = A_{i+1} \cup B_{i+1}$, then $f_i(A_{i+1}) = X_i$ or $f_i(B_{i+1}) = X_i$. Then X is indecomposable.

Example (p-adic solenoids)

Let p be a prime number. Let $X_i = \mathbb{S}^1$ be a circle and let $f_i(z) = z^p$. Then the obtained inverse limit is the p-adic solenoid and it is indecomposable.

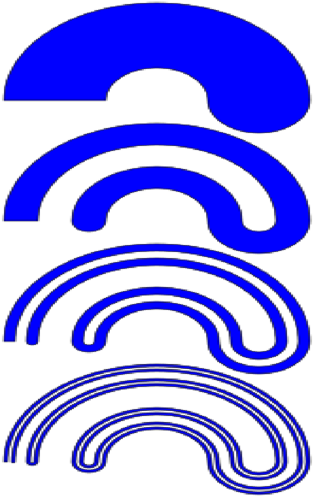
solenoid, $p = 2$



Knaster continuum is indecomposable

Let $X_i = [0,1]$
and take $X = \varprojlim\{X_i, f_i = f\}$,
where

$$f(t) = \begin{cases} 2t & \textit{for } t \in [0, \frac{1}{2}] \\ -2t + 2 & \textit{for } t \in [\frac{1}{2}, 1] \end{cases}$$



The pseudo-arc

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Theorem (Bing '48)

The pseudo-arc P is homogeneous, that is, for any $a, b \in P$ there is a homeomorphism h of P such that $h(a) = b$.

The pseudo-arc 2

Classification of topologically homogeneous plane continua:

Theorem (Hoehn-Oversteegen, 2016)

Up to homeomorphism, the only nondegenerate homogeneous planar continua are

(a) *the circle,*
(b) *the pseudo-arc, and*
(c) *the circle of pseudo-arcs.*

Projective Fraïssé theory – setup

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2. A topological L-structure is a compact zero-dimensional second-countable space A equipped with closed relations $R_i^A, i \in I$ and continuous functions $f_j^A, j \in J$.
3. Epimorphisms are continuous surjections preserving the structure.

Projective Fraïssé class – definition

A family $\mathcal{F}$ of finite topological L-structure is a projective Fraïssé class if:

1. (F1) (joint projection property: JPP) for any $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ and epimorphisms from C onto A and from C onto B;
2. (F2) (amalgamation property: AP) for $A, B_1, B_2 \in \mathcal{F}$ and any epimorphisms $\phi_1 \colon B_1 \to A$ and $\phi_2 \colon B_2 \to A$, there exist C, $\phi_3 \colon C \to B_1$ and $\phi_4 \colon C \to B_2$ such that $\phi_1 \circ \phi_3 = \phi_2 \circ \phi_4$.

Projective Fraïssé limit – definition

A topological L-structure $\mathbb{L}$ is a projective Fraïssé limit of $\mathcal{F}$ if the following three conditions hold:

1. (L1) (projective universality) for any $A \in \mathcal{F}$ there is an epimorphism from $\mathbb{L}$ onto A;
2. (L2) (projective ultrahomogeneity) for any $A \in \mathcal{F}$ and any epimorphisms $\phi_1 \colon \mathbb{L} \to A$ and $\phi_2 \colon \mathbb{L} \to A$ there exists an isomorphism $h \colon \mathbb{L} \to \mathbb{L}$ such that $\phi_2 = \phi_1 \circ h$;
3. (L3) for any finite discrete topological space X and any continuous function $f \colon \mathbb{L} \to X$ there is an $A \in \mathcal{F}$, an epimorphism $\phi \colon \mathbb{L} \to A$, and a function $f_0 \colon A \to X$ such that $f = f_0 \circ \phi$.

Projective Fraïssé limit – existence and uniqueness

Theorem (Irwin-Solecki)

Let $\mathcal{F}$ be a countable projective Fraïssé class of finite structures. Then:

1. *there exists a projective Fraïssé limit of $\mathcal{F}$;*
2. *any two projective Fraïssé limits are isomorphic.*

A simple example of a projective Fraïssé class

Let $\mathcal{F}$ be the family of all finite sets.

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Let $\mathcal{F}$ be the family of all finite sets.

Then the projective Fraïssé limit is the Cantor set.

One more simple example

Let $\mathcal{F}$ be the family of all finite sets $A = \{a_1, \ldots, a_n\}$, some n, with the binary relation $\leq^A$, where for each i, $a_i \leq^A a_i$ and $a_i \leq^A a_{i+1}$.

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Then the projective Fraïssé limit is $(\mathbb{C}, \leq^{\mathbb{C}})$, where $\mathbb{C}$ is the Cantor set. For $a \neq b \in \mathbb{C}$, we have $a \leq^{\mathbb{C}} b$ or $b \leq^{\mathbb{C}} a$ iff a and b are endpoints of an interval removed at some stage of the construction of $\mathbb{C}$, viewed as the middle-third Cantor set.

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Identify $\leq^{\mathbb{C}}$-related points. This is the *topological realization* of $(\mathbb{C}, \leq^{\mathbb{C}})$. It is homeomorphic to [0,1].

Construction of the pseudo-arc, part 1

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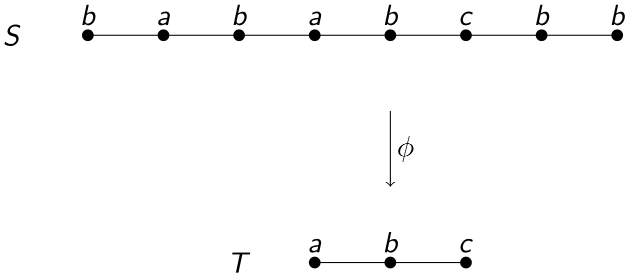
A continuous surjection $\phi\colon S \to T$ is an epimorphism iff

$$r^T(a, b) \iff \exists c, d \in S \left(\phi(c) = a, \phi(d) = b, \text{ and } r^S(c, d)\right).$$

An example of an epimorphism



Construction of the pseudo-arc, part 2

Theorem (Irwin-Solecki)

1. *The family $\mathcal{G}$ has the amalgamation property.*
2. *There is a unique $\mathbb{P} = (\mathbb{P}, r^{\mathbb{P}})$, where $\mathbb{P}$ is compact, separable, totally disconnected, $r^{\mathbb{P}}$ is closed, which is projectively universal, projectively ultrahomogeneous, and continuous maps onto finite sets factor through epimorphisms onto finite structures.*
3. *The relation $r^{\mathbb{P}}$ is an equivalence relation such that each equivalence class has at most two elements.*

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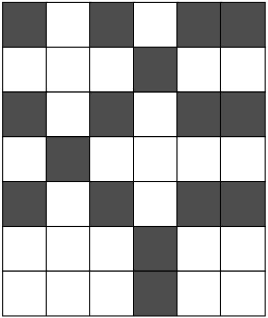
Theorem (Irwin-Solecki)

$\mathbb{P}/r^{\mathbb{P}}$ *is the pseudo-arc.*

Amalgamation property

Theorem (Steinhaus chessboard theorem)

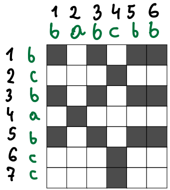
Consider a chessboard $m \times n$ with some squares black and some white. Assume that the king cannot go across the chessboard from the left edge to the right moving exclusively on black squares. Then the rook can go across the chessboard from upper edge to the lower one moving exclusively on white squares.

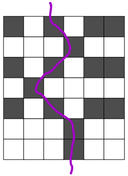


Amalgamation property 2

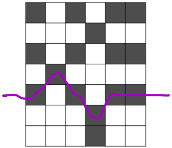
$f: \underbrace{babcbb}_{B} \to \underbrace{abc}_{A}$

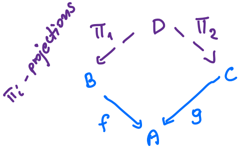
$g: \underbrace{bcbabcc}_{C} \to abc$





$\{(x,y) : f(x) = g(y)\}$ = black





D - combine both purple paths into one

D = {(1,3), (2,4), (3,3), (4,2), (5,3), (6,4), (7,4), (6,4), (5,5), (5,6), (5,5), (6,4), (5,3), (4,2), (5,1)}

Projective universality and homogeneity

The projective universality and homogeneity of $\mathbb{P}$ yield the following theorem.

Theorem

(i) *(Mioduszewski) Each chainable continuum is a continuous image of the pseudo-arc.*

(ii) *(Irwin-Solecki) Let X be a chainable continuum with a metric d on it. If f_1, f_2 are continuous surjections from the pseudo-arc onto X, then for any $\epsilon > 0$ there exists a homeomorphism h of the pseudo-arc such that $d(f_1(x), f_2 \circ h(x)) < \epsilon$ for all x.*