A tutorial on Club-Guessing
Part III

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Conventions

Throughout this series of talks:

- $\theta < \kappa$ is a pair of infinite regular cardinals;
- E^κ_θ stands for $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \theta\}$;
- S and T denote stationary subset of κ.
  Typically, they consist of limits ordinals;
- For $D \subseteq \kappa$, $\mathrm{acc}(D) := \{\delta \in D \mid \sup(D \cap \delta) = \delta > 0\}$,
  and $\mathrm{nacc}(D) := D \setminus \mathrm{acc}(D)$.

Some variations: $E^\kappa_{<\theta}, E^\kappa_{\leq\theta}, E^\kappa_{\neq\theta}, E^\kappa_{>\theta}, E^\kappa_{\geq\theta}$ and
$\mathrm{acc}^+(X) := \{\delta < \sup(X) \mid \sup(X \cap \delta) = \delta > 0\}$.

Recall

Definition

CG(S) asserts the existence of a C-sequence $\langle C_\delta \mid \delta \in S\rangle$ such that for every club $D \subseteq \kappa$, there a exists a $\delta \in S$ such that $C_\delta \subseteq D$.
Equivalently, $\{\delta \in S \mid C_\delta \subseteq D \cap \delta\}$ is stationary in κ.

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Theorem (Shelah)

CG(E^κ_θ) *holds, provided that* $\theta^+ < \kappa$.

Theorem (Abraham-Shelah)

CG($E^{\theta^+}_\theta$) *may consistently fail.*

Recall

Definition

CG(S) asserts the existence of a C-sequence $\langle C_\delta \mid \delta \in S\rangle$ such that for every club $D \subseteq \kappa$, there a exists a $\delta \in S$ such that $C_\delta \subseteq D$.

The invasion of ideals

Suppose $\vec{J} = \langle J_\delta \mid \delta \in S\rangle$ is a sequence such that each J_δ is some ideal over δ, extending $J^{\mathrm{bd}}[\delta]$ (the ideal of bounded subsets of δ).
CG($S, \vec{J}$) asserts the existence of a C-sequence $\langle C_\delta \mid \delta \in S\rangle$ such that for every club $D \subseteq \kappa$, there exists a $\delta \in S$ for which

$$\{\beta < \delta \mid \min(C_\delta \setminus (\beta+1)) \in D\} \in J_\delta^+.$$

Recall

Theorem (Shelah)

If $\kappa \geq \aleph_2$, then for every stationary $S \subseteq \kappa$, $\mathrm{CG}(S, \langle J^{\mathrm{bd}}[\delta] \mid \delta \in S \rangle)$ holds.

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Incorporating with relative club-guessing

Suppose $\vec{J} = \langle J_\delta \mid \delta \in S\rangle$ is a sequence such that each J_δ is some ideal over δ, extending $J^{\mathrm{bd}}[\delta]$ (the ideal of bounded subsets of δ). $\mathrm{CG}(S, T, \vec{J})$ asserts the existence of a C-sequence $\langle C_\delta \mid \delta \in S\rangle$ such that for every club $D \subseteq \kappa$, there exists a $\delta \in S$ for which

$$\{\beta < \delta \mid \min(C_\delta \setminus (\beta + 1)) \in D \cap T\} \in J_\delta^+.$$

Using weaker forms of club-guessing

In our ZFC Sierpiński-type theorem for singular cardinals, we used the fact that $\mathrm{CG}(E^{\kappa}_{\theta})$ allows to generate whole of θ upon any cofinal subset of κ. Is it possible to obtain a similar effect from the idealized form of guessing $\mathrm{CG}(E^{\kappa}_{\theta}, T, \langle J^{\mathrm{bd}}[\delta] \mid \delta \in S\rangle)$?

A related concept may be found in Moore's 2008 construction of an Aronszajn line with no Countryman suborders from the principle $\mho$.

Definition (Moore)

$\mho$ asserts the existence of a sequence $\langle h_\delta \mid \delta < \omega_1\rangle$ such that each h_δ is a continuous map from δ to ω, and for every club $D \subseteq \omega_1$, there is a $\delta \in D$ such that $h_\delta[D] = \omega$.

In other words, there exists a colored ω-bounded C-sequence $\langle h_\delta : C_\delta \to \omega \mid \delta \in E^{\omega_1}_{\omega}\rangle$ such that, for every club $D \subseteq \omega_1$, there exists a $\delta \in E^{\omega_1}_{\omega}$ for which $\{h_\delta(\min(C_\delta \setminus \alpha)) \mid \alpha \in D\} = \omega$.

Partitioned club-guessing

Suppose that a C-sequence $\langle C_\delta \mid \delta \in S\rangle$ witnesses $\mathsf{CG}(S, T, \vec{J})$, i.e., for every club $D \subseteq \kappa$, there exists a $\delta \in S$ such that

$$\{\beta < \delta \mid \min(C_\delta \setminus (\beta + 1)) \in D \cap T\} \in J_\delta^+.$$

We would like to find colorings $\langle h_\delta : \delta \to \mu \mid \delta \in S\rangle$ such that for every club $D \subseteq \kappa$, there exists a $\delta \in S$ such that

$$\bigwedge_{\tau<\mu} \{\beta < \delta \mid h_\delta(\beta) = \tau \ \&\ \min(C_\delta \setminus (\beta + 1)) \in D \cap T\} \in J_\delta^+.$$

Partitioned club-guessing (cont.)

Theorem ([46])

Suppose that $\langle C_\delta \mid \delta \in E^\kappa_\theta \rangle$ *witnesses* $\mathrm{CG}(S, T, \langle J^{\mathrm{bd}}[\delta] \mid \delta \in S \rangle)$. *If* $\theta \geq 2^{\aleph_0}$ *is not weakly compact, then there exist colorings* h_δ*'s such that for every club* $D \subseteq \kappa$*, there exists a* $\delta \in S$ *such that*

$$\bigwedge_{n<\omega} \sup\{\beta < \delta \mid h_\delta(\beta) = n \ \& \ \min(C_\delta \setminus (\beta+1)) \in D \cap T\} = \delta.$$

Partitioned club-guessing (cont.)

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Theorem ([46])

Suppose that $\langle C_\delta \mid \delta \in E^\kappa_\theta \rangle$ *witnesses* $\mathrm{CG}(S, T, \langle \mathrm{NS}_\delta \mid \delta \in S \rangle)$*.*
If $\theta \geq 2^{\aleph_0}$ *is not ineffable, then there exist colorings* h_δ*'s such that for every club* $D \subseteq \kappa$*, there exists a* $\delta \in S$ *such that*

$$\bigwedge_{n<\omega} \{\beta < \delta \mid h_\delta(\beta) = n \ \&\ \min(C_\delta \setminus (\beta+1)) \in D \cap T\} \in \mathrm{NS}^+_\delta.$$

Partitioned club-guessing (cont.)

Theorem ([46])

Suppose that $\langle C_\delta \mid \delta \in E^\kappa_\theta \rangle$ *witnesses* $\mathrm{CG}(S, T, \langle \mathrm{NS}_\delta \mid \delta \in S \rangle)$.
If $\diamondsuit^*(\theta)$ *holds, then there exist colorings* h_δ*'s such that for every club* $D \subseteq \kappa$*, there exists a* $\delta \in S$ *such that*

$$\bigwedge_{\tau<\theta} \{\beta < \delta \mid h_\delta(\beta) = \tau \;\&\; \min(C_\delta \setminus (\beta+1)) \in D \cap T\} \in \mathrm{NS}^+_\delta .$$

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Partitioned club-guessing (cont.)

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Often times, we can get by with hypotheses considerably weaker than $\diamondsuit^*$, using a bulk of results from [47,53].

Partitioned club-guessing (cont.)

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Suppose that $\langle C_\delta \mid \delta \in E^\kappa_\theta \rangle$ *witnesses* $\mathrm{CG}(S, T, \langle \mathrm{NS}_\delta \mid \delta \in S \rangle)$.
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Often times, we can get by with hypotheses considerably weaker than $\diamondsuit^*$, using a bulk of results from [47,53].

And sometimes it will be the combinatorial nature of κ (instead of θ) to be saving our day. . . :

Applications of abstract nonsense

Suppose θ is regular and uncountable, and $S \subseteq E^{\theta^+}_\theta$.

Given a witness $\vec{C}$ to $\mathrm{CG}(S, T, \langle J^{\mathrm{bd}}[\delta] \mid \delta \in S\rangle)$, the collection

$\mathcal{I} := \{X \subseteq \theta^+ \mid \vec{C}$ does not witness $\mathrm{CG}(S, X \cap T, \langle J^{\mathrm{bd}}[\delta] \mid \delta \in S\rangle)\}$

is a θ-additive ideal on θ^+ extending NS_{θ^+}.

Lemma ([52])

If $\square(\theta^+, {<}\theta)$ holds, then every θ-additive ideal on θ^+ extending $J^{\mathrm{bd}}[\theta^+]$ is not weakly θ-saturated.

Corollary

If $\square(\theta^+, {<}\theta)$ holds, then there is a decomposition $T = \biguplus_{i<\theta} T_i$ such that $\vec{C}$ witnesses $\mathrm{CG}(S, T_i, \langle J^{\mathrm{bd}}[\delta] \mid \delta \in S\rangle)$ for all $i < \theta$.

Now, define $h_\delta : \delta \to \theta$ by letting $h_\delta(\beta) := i$ for the unique $i < \theta$ such that $\min(C_\delta \setminus \beta) \in T_i$. (if $\min(C_\delta \setminus \beta) \notin T$, let $h_\delta(\beta) := 0$)

Conclusions

- Relaxing $\mathrm{CG}(S)$ to $\mathrm{CG}(S, \vec{J})$ is unavoidable (e.g., $S = E_\theta^{\theta^+}$);
- One can compensate for this relaxation by having a partitioned form of $\mathrm{CG}(S, \vec{J})$;
- Partitioned form of guessing can easily be deduced from the gallery of colorings obtained in [47,53], and the "better" the sequence of ideals is, the more colors one gets...
  The J_δ's being $J^{\mathrm{bd}}[\delta]$ is OK; being NS_δ is better; all the J_δ's being a copy of $\mathrm{NS}_\theta + \mathrm{Reg}(\theta)$ is ideal (pun intended);
- This calls for moving between ideals.

Moving between ideals

Suppose θ is regular and uncountable.

Theorem (Shelah)

For $S \subseteq E^{\theta^+}_\theta$ and $T := E^{\theta^+}_\theta$:

1. $\mathrm{CG}(S, T, \langle J^{\mathrm{bd}}[\delta] \mid \delta \in S\rangle)$ *holds;*
2. $\mathrm{CG}(S, \theta^+, \langle \mathrm{NS}_\delta \mid \delta \in S\rangle)$ *holds.*

In [46], we found a way to move between ideals while maintaining relative club-guessing.

Corollary ([46])

For $S \subseteq E^{\theta^+}_\theta$ and $T := E^{\theta^+}_\theta$, $\mathrm{CG}(S, T, \langle \mathrm{NS}_\delta \mid \delta \in S\rangle)$ holds.

In case moving between ideals is not satisfactory, we can do a Solovay-type decomposition theorem for club guessing.

Some more abstract nonsense

Theorem (Devlin, 1978)

If $\diamondsuit(S)$ *holds, then there is a decomposition* $S = \biguplus_{i<\kappa} S_i$ *such that* $\diamondsuit(S_i)$ *holds for all* $i < \kappa$.

Theorem ([23])

If $\clubsuit(S)$ *holds, then there is a decomposition* $S = \biguplus_{i<\kappa} S_i$ *such that* $\clubsuit(S_i)$ *holds for all* $i < \kappa$.

Theorem ([46])

If $\vec{C} = \langle C_\delta \mid \delta \in S \rangle$ *witnesses* $\mathrm{CG}(S, T, \vec{J})$, *there is* $S = \biguplus_{i<\kappa} S_i$ *such that* $\vec{C} \restriction S_i$ *witnesses* $\mathrm{CG}(S_i, T, \vec{J})$ *for all* $i < \kappa$.

This means that we may settle for maps $h_\delta : \delta \to i$ for $\delta \in S_i$ that will gradually generate all colors.

Some more abstract nonsense (cont.)

Theorem [46]

If $\vec{C} = \langle C_\delta \mid \delta \in S\rangle$ witnesses $\mathrm{CG}(S, T, \vec{J})$, there is $S = \biguplus_{i<\kappa} S_i$ such that $\vec{C} \restriction S_i$ witnesses $\mathrm{CG}(S_i, T, \vec{J})$ for all $i < \kappa$.

Proof. Denote $S_i^\beta := \{\delta \in S \cap \mathrm{acc}(\kappa \setminus \beta) \mid \min(C_\delta \setminus (\beta + 1)) = i\}$. It suffices to prove there is a $\beta < \kappa$ s.t. the following set has size κ:

$$I_\beta := \{i \in (\beta, \kappa) \mid \vec{C} \restriction S_i^\beta \text{ witnesses } \mathrm{CG}_\xi(S_i^\beta, T, \vec{J})\}.$$

So, suppose that this is not the case, and fix a sparse enough club $E \subseteq \kappa$ such that, for every $\epsilon \in E$, for every $\beta < \epsilon$, $\sup(I_\beta) < \epsilon$. In addition, fix a triangular matrix $\langle D_i^\beta \mid \beta < i < \kappa\rangle$ of clubs in κ such that, for all $\beta < i < \kappa$, if $i \notin I_\beta$, then for every $\delta \in S_i^\beta$,

$$\{\beta < \delta \mid \min(C_\delta \setminus (\beta + 1)) \in D_i^\beta \cap T\} \in J_\delta.$$

Some more abstract nonsense (cont.)

Consider the club $D := \{\delta \in E \mid \forall i < \delta \forall \beta < i(\delta \in D_i^\beta)\}$.
By the choice of $\vec{C}$, pick $\delta \in S$ such that the following set is in J_δ^+:

$$B := \{\beta < \delta \mid \min(C_\delta \setminus (\beta + 1)) \in D \cap T\}.$$

Claim

For every $\beta < \delta$, $\min(C_\delta \setminus (\beta + 1)) \in I_\beta$.

Proof. Given $\beta < \delta$, if we let $i := \min(C_\delta \setminus (\beta + 1))$, then $\delta \in S_i^\beta$, and since $D \cap \delta$ is almost included in $D_i^\beta \cap \delta$, it is the case that

$$\{\beta < \delta \mid \min(C_\delta \setminus (\beta + 1)) \in D_i^\beta \cap T\} \in J_\delta^+,$$

so that $i \in I_\beta$. □

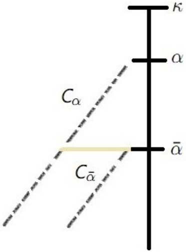
Pick $\beta \in B$ and set $\epsilon := \min(C_\delta \setminus (\beta + 1))$. As $\epsilon \in D \cap T \subseteq E$ and $\beta < \epsilon$, $\sup(I_\beta) < \epsilon$, contradicting the preceding claim. □

Moving the S

Definition

A C-sequence $\vec{C} = \langle C_\delta \mid \delta < \kappa \rangle$ is coherent if for all $\bar{\alpha} < \alpha < \kappa$,

$$\bar{\alpha} \in \mathrm{acc}(C_\alpha) \text{ iff } C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}.$$



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Recall

$\Phi : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ is a postprocessing function if for all $x \in \mathcal{K}(\kappa)$:

1. $\Phi(x)$ is a club in $\sup(x)$;
2. $\mathrm{acc}(\Phi(x)) \subseteq \mathrm{acc}(x)$;
3. $\Phi(x) \cap \bar{\alpha} = \Phi(x \cap \bar{\alpha})$ for every $\bar{\alpha} \in \mathrm{acc}(\Phi(x))$.

Clause (3) implies that if $\vec{C}$ is coherent, then so is $\langle \Phi(C_\delta) \mid \delta < \kappa \rangle$.

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A C-sequence $\vec{C} = \langle C_\delta \mid \delta < \kappa \rangle$ is coherent if for all $\bar{\alpha} < \alpha < \kappa$,

$$\bar{\alpha} \in \mathrm{acc}(C_\alpha) \text{ iff } C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}.$$

Theorem ([46])

Suppose that $\kappa \geq \aleph_2$.
If there exists a coherent witness to $\mathrm{CG}(\kappa, T, \langle J^{\mathrm{bd}}[\delta] \mid \delta < \kappa \rangle)$, then $\mathrm{CG}(S, T, \langle J^{\mathrm{bd}}[\delta] \mid \delta \in S \rangle)$ holds for every stationary $S \subseteq \kappa$.

Moving the T

The regressive functions ideal ([24])

$J[\kappa]$ stands for the collection of all subsets $S \subseteq \kappa$ for which there exist a club $D \subseteq \kappa$ and a sequence of functions $\langle f_i : \kappa \to \kappa \mid i < \kappa \rangle$ with the property that for every $\delta \in S \cap D$, every regressive map $f : \delta \to \delta$, and every cofinal subset $\Gamma \subseteq \delta$, there is an $i < \delta$ s.t.

$$\sup\{\gamma \in \Gamma \mid f(\gamma) = f_i(\gamma)\} = \delta.$$

Theorem ([46])

Suppose that $\mathrm{CG}(S, \kappa, \vec{J})$ *holds, and* $S \in J[\kappa]$.
Then $\mathrm{CG}(S, T, \vec{J})$ *holds for every stationary* $T \subseteq \kappa$.

The proof takes a witness to $\mathrm{CG}(S, \kappa, \vec{J})$ and applies a postprocessing function to get $\mathrm{CG}(S, T, \vec{J})$.

Moving the T

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Moving the T

Theorem ([51])

1. $J[\omega_1]$ *contains no stationary sets;*
2. $J[\omega_2]$ *contains a stationary set iff there is a nonmeager set of reals of size* $\aleph_1$*;*
3. $J[\lambda^+]$ *contains a stationary set for every* $\lambda > 2^{\aleph_1}$ *provided that Shelah's Strong Hypothesis holds;*
4. $J[\lambda^+]$ *contains a stationary set for every* $\lambda \geq \beth_\omega$*.*

Theorem ([46])

Suppose that $\mathrm{CG}(S, \kappa, \vec{J})$ *holds, and* $S \in J[\kappa]$*.*
Then $\mathrm{CG}(S, T, \vec{J})$ *holds for every stationary* $T \subseteq \kappa$*.*

The proof takes a witness to $\mathrm{CG}(S, \kappa, \vec{J})$ and applies a postprocessing function to get $\mathrm{CG}(S, T, \vec{J})$.

Putting it all together

Corollary ([46])

Suppose $\lambda \geq \beth_\omega$ is such that $\square(\lambda^+)$ holds, and S, T are stationary subsets of λ^+. Then $\mathsf{CG}(S, T, \langle J^{\mathsf{bd}}[\delta] \mid \delta \in S\rangle)$ holds.

Proof. (1) As $\lambda \geq \beth_\omega$, we may fix a stationary set $R \in J[\lambda^+]$.
(2) By [29], a witness to $\square(\lambda^+)$ is an amenable C-sequence, so by invoking the postprocessing Φ_D^{drop} from yesterday, we obtain a coherent witness to $\mathsf{CG}(R, \kappa, \langle J^{\mathsf{bd}}[\delta] \mid \delta \in R\rangle)$.
(3) It now follows from the result of the previous slide (*moving the T*) that there is a coherent witness to $\mathsf{CG}(R, T, \langle J^{\mathsf{bd}}[\delta] \mid \delta \in R\rangle)$.
In fact, there is a coherent witness to $\mathsf{CG}(\kappa, T, \langle J^{\mathsf{bd}}[\delta] \mid \delta < \kappa\rangle)$.
(4) It now follows from the result of the previous-previous slide (*moving the S*) that $\mathsf{CG}(S, T, \langle J^{\mathsf{bd}}[\delta] \mid \delta \in S\rangle)$ holds. □

Some open problems

Question (Moving between ideals)

Suppose that S consists of ineffables. Does $\mathrm{CG}(S,\kappa,\vec{J})$ hold where each J_δ is $\mathrm{NS}_\delta \restriction R_\delta$, for some small stationary subset R_δ of δ, supporting an amenable C-sequence?

Question

Suppose that λ is a singular cardinal of coutnable cofinality. Must there exist a C-sequence $\langle C_\delta \mid \delta \in E^{\lambda^+}_{\mathrm{cf}(\lambda)}\rangle$ such that for every club $D \subseteq \lambda^+$, there exists a $\delta \in E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ satisfying the following?

1. $C_\delta \subseteq D$;
2. $\langle \mathrm{cf}(\beta) \mid \beta \in C_\delta\rangle$ is an increasing sequence converging to λ.

An affirmative answer would follow from a Q we asked yesterday:

Question

Suppose that λ is a singular cardinal. Must there exist a C-sequence $\langle C_\delta \mid \delta \in E^{\lambda^+}_{\mathrm{cf}(\lambda)}\rangle$ such that for every club $D \subseteq \lambda^+$, the set $\{\delta \in E^{\lambda^+}_{\mathrm{cf}(\lambda)} \mid C_\delta \subseteq D \;\&\; \mathrm{otp}(C_\delta) = \lambda\}$ is stationary?