A tutorial on Club-Guessing
Part II

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Conventions

Throughout this series of talks:

- $\theta < \kappa$ is a pair of infinite regular cardinals;
- E^κ_θ stands for $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \theta\}$;
- S denotes a stationary subset of κ.
  Typically, S consists of limits ordinals;
- For $D \subseteq \kappa$, $\mathrm{acc}(D) := \{\delta \in D \mid \sup(D \cap \delta) = \delta > 0\}$,
  and $\mathrm{nacc}(D) := D \setminus \mathrm{acc}(D)$.

Some variations: $E^\kappa_{<\theta}, E^\kappa_{\leq\theta}, E^\kappa_{\neq\theta}, E^\kappa_{>\theta}, E^\kappa_{\geq\theta}$ and
$\mathrm{acc}^+(X) := \{\delta < \sup(X) \mid \sup(X \cap \delta) = \delta > 0\}$.

Recall I

Definition ([29])

$\Phi : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ is a postprocessing function if for all $x \in \mathcal{K}(\kappa)$:

1. $\Phi(x)$ is a club in $\sup(x)$;
2. $\mathrm{acc}(\Phi(x)) \subseteq \mathrm{acc}(x)$;
3. $\Phi(x) \cap \bar{\alpha} = \Phi(x \cap \bar{\alpha})$ for every $\bar{\alpha} \in \mathrm{acc}(\Phi(x))$.

It is conservative if $\Phi(x) \subseteq x$ for all x.

Recall II

Definition (Shelah, 1990's)

CG(S) asserts the existence of a sequence $\vec{C} = \langle C_\delta \mid \delta \in S \rangle$ s.t.:

1. for every $\delta \in S$, C_δ is a club in δ;
2. for every club $D \subseteq \kappa$, the set $\{\delta \in S \mid C_\delta \subseteq D \cap \delta\}$ is stationary in κ.

Theorem (Shelah)

CG(E^κ_θ) *holds, in any of the following cases:*

- $\aleph_0 < \theta < \theta^+ < \kappa$;
- $\aleph_0 = \theta$ *and* $\kappa = \lambda^+$ *for some uncountable cardinal* λ.

Sierpiński-type colorings

Definition ([47])

Onto$(\lambda, \kappa, \theta)$ asserts the existence of a coloring $c : \lambda \times \kappa \to \theta$ such that, for every $B \in [\kappa]^\kappa$, for some $\eta < \lambda$, $c[\{\eta\} \times B] = \theta$.

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Fact ([53])

If $\mathrm{Onto}(\lambda, \kappa, \theta)$ *holds, then there is a λ-sized universal family of decompositions of κ into θ many sets, $\{\langle U_{\eta,\tau} \mid \tau < \theta\rangle \mid \eta < \lambda\}$. This means that for every λ^+-complete ideal J extending $[\kappa]^{<\kappa}$, for every $B \in J^+$, for some $\eta < \lambda$,*

$$\langle U_{\eta,\tau} \cap B \mid \tau < \theta\rangle$$

is a decomposition of B into θ many J^+-sets.

This is actually an equivalency in the non-degenerate case.

Sierpiński-type colorings

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Sierpiński proved that Onto$(\lambda, \lambda^+, \lambda^+)$ follows from $2^\lambda = \lambda^+$, and we mentioned yesterday that Onto$(\lambda, \lambda^+, \lambda)$ may consistently fail. This leaves open the case Onto$(\lambda, \lambda^+, \theta)$ for $\theta < \lambda$.

A Sierpiński theorem in ZFC

Theorem ([53])

Suppose that λ is a singular cardinal. For every cardinal $\theta < \lambda$, there is a coloring $c : \lambda \times \lambda^+ \to \theta$ such that, for every $B \in [\lambda^+]^{\lambda^+}$, for some $\eta < \lambda$, $c[\{\eta\} \circledast B] = \theta$. I.e., $\mathrm{Onto}(\lambda, \lambda^+, \theta)$ holds.

For the proof, we need Shelah's theorem on the existence of scales.

Fact (Shelah)

For every singular cardinal λ, there is a sequence $\langle f_\beta \mid \beta < \lambda^+ \rangle$:

- *All f_β's are elements of some product $\prod_{i<\mathrm{cf}(\lambda)} \lambda_i$ of regular cardinals, with $\sup\{\lambda_i \mid i < \mathrm{cf}(\lambda)\} = \lambda$;*
- *For all $\alpha < \beta < \lambda^+$, $f_\alpha <^* f_\beta$, that is, $f_\alpha(i) < f_\beta(i)$ for a tail of $i < \mathrm{cf}(\lambda)$;*
- *For every $g \in \prod_{i<\mathrm{cf}(\lambda)} \lambda_i$, there is $\beta < \lambda^+$ such that $g <^* f_\beta$.*

A Sierpiński-type theorem in ZFC

Theorem ([53])

Suppose that λ is a singular cardinal. For every cardinal $\theta < \lambda$, there is a coloring $c : \lambda \times \lambda^+ \to \theta$ such that, for every $B \in [\lambda^+]^{\lambda^+}$, for some $\eta < \lambda$, $c[\{\eta\} \times B] = \theta$.

Proof. Fix a scale $\langle f_\beta \mid \beta < \lambda^+ \rangle$ in some product $\prod_{i<\mathrm{cf}(\lambda)} \lambda_i$.
By increasing θ and λ_0, may assume $\mathrm{cf}(\lambda) < \mathrm{cf}(\theta) = \theta < \theta^+ < \lambda_0$.
For $i < \mathrm{cf}(\lambda)$, fix a θ-bounded witness $\langle C^i_\delta \mid \delta \in E^{\lambda_i}_\theta \rangle$ to $\mathrm{CG}(E^{\lambda_i}_\theta)$.
Fix a bijection $\pi : \lambda \leftrightarrow \bigcup_{i<\mathrm{cf}(\lambda)}(\{i\} \times E^{\lambda_i}_\theta)$. Pick $c : \lambda \times \lambda^+ \to \theta$ such that, for all $\eta < \lambda$ and $\beta < \lambda^+$, if $\pi(\eta) = (i, \delta)$, then

$$c(\eta, \beta) := \sup(\mathrm{otp}(C^i_\delta \cap f_\beta(i))).$$

To see this works, let $B \in [\lambda^+]^{\lambda^+}$. Then $\langle f_\beta \mid \beta \in B \rangle$ is a scale, so there must exist an $i < \mathrm{cf}(\lambda)$ such that $\sup\{f_\beta(i) \mid \beta \in B\} = \lambda_i$.
As $D := \mathrm{acc}^+(\{f_\beta(i) \mid \beta \in B\})$ is a club in λ_i, we may fix some $\delta \in E^{\lambda_i}_\theta$ such that $C^i_\delta \subseteq D$. Set $\eta := \pi^{-1}(i, \delta)$.
In between any two elements of C^i_δ, there is one the form $f_\beta(i)$ for some $\beta \in B$. So $c[\{\eta\} \circledast B] = \theta$, as sought! □

The critical cofinality

Theorem (Abraham-Shelah, [AbSh:182])

Assume GCH and $\kappa = \theta^+$. (recall that θ is assumed to be regular) Then there is a GCH-preserving forcing extension, adding no new θ-sequences, not collapsing cardinals, in which $\mathrm{CG}(E^{\theta^+}_\theta)$ fails. Furthermore, in this model, there is a family $\langle D_i \mid i < \kappa^+ \rangle$ of clubs in κ such that $|\bigcap_{i \in I} D_i| < \theta$ for every $I \in [\kappa^+]^\kappa$.

The analogous question for successors of singulars is open.
To focus on the contrapositive:

Question

Suppose that λ is a singular cardinal. Must there exist a C-sequence $\langle C_\delta \mid \delta \in E^{\lambda^+}_{\mathrm{cf}(\lambda)} \rangle$ such that for every club $D \subseteq \lambda^+$, the set $\{\delta \in E^{\lambda^+}_{\mathrm{cf}(\lambda)} \mid C_\delta \subseteq D \;\&\; \mathrm{otp}(C_\delta) = \lambda\}$ is stationary?

The second theorem

Theorem (Shelah)

For every regular uncountable cardinal θ, there exists a θ-bounded C-sequence $\langle C_\delta \mid \delta \in E^{\theta^+}_\theta \rangle$ such that, for every club $D \subseteq \theta^+$, the following set is stationary:

$$\{\delta \in E^{\theta^+}_\theta \mid \sup(\mathrm{nacc}(C_\delta) \cap D) = \delta\}.$$

Equivalently: for every club $D \subseteq \theta^+$, the next set is nonempty:

$$\{\delta \in E^{\theta^+}_\theta \mid \sup\{\beta < \delta \mid \min(C_\delta \setminus (\beta + 1)) \in D\} = \delta\}.$$

The invasion of ideals

Suppose $\vec{J} = \langle J_\delta \mid \delta \in S \rangle$ is a sequence such that each J_δ is some ideal over δ, extending $J^{\mathsf{bd}}[\delta]$ (the ideal of bounded subsets of δ).

Definition ([46])

$\mathsf{CG}(S, \vec{J})$ asserts the existence of a C-sequence $\langle C_\delta \mid \delta \in S \rangle$ such that for every club $D \subseteq \kappa$, there exists a $\delta \in S$ for which

$$\{\beta < \delta \mid \min(C_\delta \setminus (\beta + 1)) \in D\} \in J_\delta^+.$$

Remark

So far we obtained instances of $\mathsf{CG}(S)$ by starting with some C-sequence $\langle C_\delta \mid \delta \in S \rangle$ for which $\sup_{\delta \in S} |C_\delta|$ is relatively small, and then rectifying errors using a postprocessing function.
In the context in which $\sup_{\delta \in S} |C_\delta|$ cannot be small, we need a more relaxed concept...

Amenable C-sequences

Definition ([29])

A C-sequence $\langle C_\delta \mid \delta \in S\rangle$ is amenable iff for every club $D \subseteq \kappa$, the set $\{\delta \in S \mid \sup(D \cap \delta \setminus C_\delta) < \delta\}$ is nonstationary (in κ).

Lemma

Every successor cardinal admits an amenable C-sequence.

Proof. Let $\kappa = \lambda^+$ be some successor cardinal. Pick a C-sequence $\vec{C} = \langle C_\delta \mid \delta < \kappa\rangle$ such that $\mathrm{otp}(C_\delta) \leq \lambda$ for all $\delta < \kappa$.
For every club $D \subseteq \kappa$, the set $\{\delta < \kappa \mid \mathrm{otp}(D \cap \delta) = \delta > \lambda\}$ is a club in κ disjoint from $\{\delta \in S \mid \sup(D \cap \delta \setminus C_\delta) < \delta\}$. □

More generally, if almost all ordinals in S are singular, then S admits an amenable C-sequence.

Exercise: $\langle C_\delta \mid \delta \in S\rangle$ is amenable iff for every club $D \subseteq \kappa$, the set $\{\delta \in S \mid D \cap \delta \subseteq C_\delta\}$ is nonstationary iff for every club $D \subseteq \kappa$ and every conservative Φ, $\{\delta \in S \mid D \cap \delta = \Phi(C_\delta)\}$ is nonstationary.

Amenable C-sequences

Definition

A C-sequence $\langle C_\delta \mid \delta \in S\rangle$ is amenable iff for every club $D \subseteq \kappa$, the set $\{\delta \in S \mid D \cap \delta \subseteq C_\delta\}$ is nonstationary in κ.

Lemma

For every stationary $S \subseteq \kappa$, there exists a stationary $S' \subseteq S$ such that S' carries an amenable C-sequence.

Proof. If $S \cap E^\kappa_\omega$ is stationary, then $S' := S \cap E^\kappa_\omega$ carries an ω-bounded C-sequence, which is clearly amenable. So, we may assume that $S \cap E^\kappa_\omega$ is empty, and let $S' := S \setminus \mathrm{Tr}(S)$, where:

$$\mathrm{Tr}(S) := \{\alpha \in E^\kappa_{>\omega} \mid S \cap \alpha \text{ is stationary in } \alpha\},$$

▶ To see that S' is stationary, let $D \subseteq \kappa$ be a club. Then $\alpha := \min(\mathrm{acc}(D) \cap S)$ belongs to $D \cap S$ and $\mathrm{acc}(D) \cap \alpha$ is a club in α disjoint from S, so that $\alpha \notin \mathrm{Tr}(S)$. Altogether, $\alpha \in S' \cap D$. ◀

Fix a C-sequence $\langle C_\delta \mid \delta \in S'\rangle$ such that $C_\delta \cap S = \emptyset$ for all $\delta \in S'$. If there is a club $D \subseteq \kappa$ such that $\{\delta \in S' \mid D \cap \delta \subseteq C_\delta\}$ is cofinal in κ, then that set must contain a $\delta \in S'$ above $\min(D \cap S)$. □

Amenable C-sequences and club-guessing

We have seen that for every stationary $S \subseteq \kappa$, there exists a stationary $S' \subseteq S$ such that S' carries an amenable C-sequence. The next step is to come up with a postprocessing function that can take advantage of amenability. For a club $D \subseteq \kappa$, let

$$\Phi_D^{\mathrm{drop}}(x) := \begin{cases} \{\sup(D \cap \gamma) \mid \gamma \in x, \gamma > \min(D)\}, & \sup(x) \in \mathrm{acc}(D); \\ x \setminus \sup(D \cap \sup(x)), & \text{otherwise.} \end{cases}$$

Note: Φ is not conservative! $\sup(x) \in \mathrm{acc}(D) \implies \Phi_D^{\mathrm{drop}}(x) \subseteq D$.

Lemma

Suppose that $\vec{C} = \langle C_\delta \mid \delta \in S \rangle$ is an amenable C-sequence. If $\kappa \geq \aleph_2$, then there exists a club $D \subseteq \kappa$ for which $\langle \Phi_D^{drop}(C_\delta) \mid \delta \in S \rangle$ witnesses $\mathrm{CG}(S, \langle J^{\mathrm{bd}}[\delta] \mid \delta \in S \rangle)$.

Amenable C-sequences and club-guessing (cont.)

Proof. Suppose not. So, for every club $D \subseteq \kappa$, there is a club $F^D \subseteq \kappa$ such that, for every $\delta \in S$,

$$\sup(\mathrm{nacc}(\Phi^{\mathrm{drop}}_D(C_\delta)) \cap F^D) < \delta.$$

Construct a descending sequence $\langle D_i \mid i < \omega_1 \rangle$ of clubs in κ via:

1. $D_0 := \kappa$;
2. $D_{i+1} := D_i \cap F^{D_i}$;
3. for $i \in \mathrm{acc}(\omega_1)$, $D_i := \bigcap_{i'<i} D_{i'}$.

As $D^* := \bigcap_{i<\omega_1} D_i$ is a club in $\kappa \geq \aleph_2$ and $\vec{C}$ is amenable, we may pick some $\delta \in S$ such that $\sup(D^* \cap \delta \setminus C_\delta) = \delta$. For each $i < \omega_1$, since $D_i \cap \delta$ is a closed unbounded subset of δ, it is the case that

$$\Phi^{\mathrm{drop}}_{D_i}(C_\delta) = \{\sup(D_i \cap \gamma) \mid \gamma \in C_\delta, \gamma > \min(D_i)\}.$$

So $\Phi^{\mathrm{drop}}_{D_i}(C_\delta) \subseteq D_i$ and $\mathrm{acc}(\Phi^{\mathrm{drop}}_{D_i}(C_\delta)) \subseteq \mathrm{acc}(D_i) \cap \mathrm{acc}(C_\delta)$. In addition, for each $i < \omega_1$, since $D_{i+1} \subseteq F^{D_i}$,

$$\varepsilon_i := \sup(\mathrm{nacc}(\Phi^{\mathrm{drop}}_{D_i}(C_\delta)) \cap D_{i+1}) \text{ is smaller than } \delta.$$

Amenable C-sequences and club-guessing (cont.)

Claim

There exists $I \subseteq \omega_1$ of ordertype ω such that $\sup\{\varepsilon_i \mid i \in I\} < \delta$*.*

Proof. ▶ If $\mathrm{cf}(\delta) > \omega$, then just let $I := \omega$.

▶ If $\mathrm{cf}(\delta) = \omega$, then pick a countable cofinal subset e of δ and for each $i \in \omega_1$, find the least $\varepsilon \in e$ such $\varepsilon_i \leq \varepsilon$. By the pigeonhole principle, there is an $\varepsilon \in e$ for which $\{i \in I \mid \varepsilon_i \leq \varepsilon\}$ is uncountable. In particular, this set contains a subset of type ω. □

Amenable C-sequences and club-guessing (cont.)

Fix $I \subseteq \omega_1$ of ordertype ω such that $\sup\{\varepsilon_i \mid i \in I\} < \delta$, and then pick $\alpha \in D^* \cap \delta \setminus C_\delta$ above $\sup\{\varepsilon_i \mid i \in I\}$.
As $\alpha \notin C_\delta$, $\gamma := \min(C_\delta \setminus \alpha)$ is in $\mathrm{nacc}(C_\delta)$.
As $\langle \sup(D_i \cap \gamma) \mid i \in I\rangle$ is a weakly decreasing sequence of ordinals, by well-foundedness there must be a pair of ordinals $i < j$ in I such that $\beta_i := \sup(D_i \cap \gamma)$ is equal to $\beta_j := \sup(D_j \cap \gamma)$.
As $\alpha \in D_i \cap \gamma$, $\varepsilon_i < \alpha \leq \beta_i \leq \gamma$, so $\beta_i \in \Phi^{\mathrm{drop}}_{D_i}(C_\delta) \cap (\varepsilon_i, \gamma]$.
Likewise, $\beta_j \in \Phi^{\mathrm{drop}}_{D_j}(C_\delta) \cap (\varepsilon_j, \gamma]$.
Recalling that $\beta_i = \beta_j \in D_j \subseteq D_{i+1}$, it follows that β_i is an element of $\Phi^{\mathrm{drop}}_{D_i}(C_\delta) \cap D_{i+1}$ above ε_i and hence $\beta_i \in \mathrm{acc}(\Phi^{\mathrm{drop}}_{D_i}(C_\delta))$.
Recalling that $\mathrm{acc}(\Phi^{\mathrm{drop}}_{D_i}(C_\delta)) \subseteq \mathrm{acc}(D_i) \cap \mathrm{acc}(C_\delta)$, we infer that $\beta_i \in \mathrm{acc}(C_\delta)$. But $\alpha \leq \beta_i \leq \gamma$ and $C_\delta \cap [\alpha, \gamma] = \{\gamma\}$, and hence $\beta_i = \gamma$, contradicting the fact that $\gamma \in \mathrm{nacc}(C_\delta)$. □

Amenability FTW

Corollary

If $\kappa \geq \aleph_2$, then for every stationary $S \subseteq \kappa$, $\mathsf{CG}(S, \langle J^{\mathsf{bd}}[\delta] \mid \delta \in S\rangle)$ holds.

Proof. Given a stationary set S, find a stationary $S' \subseteq S$ and an amenable C-sequence $\langle C_\delta \mid \delta \in S'\rangle$. Now, find a club $D \subseteq \kappa$ such that $\langle \Phi^{\mathsf{drop}}_D(C_\delta) \mid \delta \in S'\rangle$ witnesses $\mathsf{CG}(S', \langle J^{\mathsf{bd}}[\delta] \mid \delta \in S'\rangle)$.
In particular, $\mathsf{CG}(S, \langle J^{\mathsf{bd}}[\delta] \mid \delta \in S\rangle)$ holds. $\square$

Corollary

$\mathsf{CG}(\mathrm{Reg}(\kappa), \langle J^{\mathsf{bd}}[\delta] \mid \delta \in \mathrm{Reg}(\kappa)\rangle)$ *holds for every Mahlo κ.* $\square$

A bonus

Corollary

If $\theta^+ < \kappa$*, then* $\mathsf{CG}(S)$ *holds for every stationary* $S \subseteq E^\kappa_\theta$*.*

Proof. Yesterday we took care of the case $\aleph_0 < \theta$, so suppose $S \subseteq E^\kappa_\omega$. Since postprocessing functions do not increase order-types, the result from the previous slide yields an ω-bounded witness to $\mathsf{CG}(S, \langle J^{\mathrm{bd}}[\delta] \mid \delta \in S\rangle)$.

So, by the so-called *familiar argument*, we may find a club $D \subseteq \kappa$ such that $\langle \Phi_D(C_\delta) \mid \delta \in S\rangle$ witnesses $\mathsf{CG}(S)$. □