Boolean valued semantics for infinitary logic

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Let L be a signature. Formulas in first order logic are obtained by induction with the following rules:

- atomic formulas,
- $\phi \to \neg\phi$,
- ϕ, $\psi \to \phi \wedge \psi$, $\phi \vee \psi$,
- $\phi \to \exists v \phi, \forall v \phi$.

Let κ be a regular cardinal, the rules of formation for the logic $\mathrm{L}_{\kappa\omega}$ are:

- atomic formulas,
- $\phi \to \neg\phi$,
- ~~$\phi, \psi \to \phi \wedge \psi$,~~ $\rightsquigarrow$ $\{\phi_\alpha : \alpha < \gamma\}, \gamma < \kappa, \to \bigwedge_{\alpha<\gamma} \phi_\alpha$,
- $\phi \to \exists v\phi, \forall v\phi$.

Let $\lambda \leq \kappa$ be regular cardinals, the rules of formation for the logic $\mathrm{L}_{\kappa\lambda}$ are:

- atomic formulas,
- $\phi \to \neg\phi$,
- ~~$\phi, \psi \to \phi \wedge \psi$,~~ $\rightsquigarrow$ $\{\phi_\alpha : \alpha < \gamma\} \to \bigwedge_{\alpha<\gamma} \phi_\alpha, \bigvee_{\alpha<\gamma} \phi_\alpha$
- ~~$\phi \to \forall v\phi, \exists v\phi$,~~ $\rightsquigarrow$ $\phi \to \forall \overline{v}\phi,\ \exists \overline{v}\phi$.

What can we say in $\mathrm{L}_{\omega_1\omega}$?

Connected graph

$$\forall v \forall w \left(\bigvee_{n\in\omega} \exists v_1 \ldots \exists v_n (Rvv_1 \wedge \bigwedge_{i=1}^{n} Rv_i v_{i+1} \wedge Rv_n w) \right)$$

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Finitely generated group

$$\bigvee_{n\in\omega} \left(\exists v_1 \ldots \exists v_n \forall x \bigvee_{\{i_1,\ldots,1_p\}\subset[1,n]} \bigvee_{\{m_1,\ldots,m_p\}\subset\omega} x = m_1 v_{i_1} \ * \ \ldots \ * \ m_p v_{i_p} \right)$$

What can we say in $\mathrm{L}_{\omega_1\omega_1}$?

Well founded relation

$$\neg\left(\exists\{v_n : n\in\omega\}\bigwedge_{n\in\omega} R v_{n+1} v_n\right)$$

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$\mathrm{L}_{\kappa\kappa}$-elementary embeddings

$$j : V \to \mathrm{Ult}(V, G)$$

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Thus, compactness fails even for $\mathrm{L}_{\omega_1\omega}$.

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- Fix a coutable set C,
- $\phi = \forall v \psi(v) \quad \rightsquigarrow \quad \psi(c)$ for any $c \in C$,
- $\phi = \psi \vee \theta \quad \rightsquigarrow \quad \psi$ or θ.

Definition

A consistency property is a set S of sets of $\mathrm{L}_{\kappa\lambda}$-formulas such that:

1. for any ϕ, $\phi \notin s$ or $\neg\phi \notin s$,
2. if $\bigwedge \Phi \in s$, then $s \cup \{\phi\} \in S$ for any $\phi \in \Phi$,
3. if $\bigvee \Phi \in s$, then $s \cup \{\phi\} \in S$ for some $\phi \in \Phi$,
4. if $\forall \overline{v}\phi(\overline{v}) \in s$, then $s \cup \{\phi(\overline{c})\} \in S$ for any $\overline{c} \in C^{|\overline{v}|}$,
5. if $\exists \overline{v}\phi(\overline{v}) \in s$, then $s \cup \{\phi(\overline{c})\} \in S$ for some $\overline{c} \in C^{|\overline{v}|}$.

Theorem (Model Existence Theorem, Makkai 1969)

Let L be a countable signature. If S is a consistency property for $\mathrm{L}_{\omega_1\omega}$ whose elements are all countable, then any $s \in S$ is consistent.

Corollary

A sentence $\phi \in \mathrm{L}_{\omega_1\omega}$ has a model if and only if there exists a consistency propety S and some $s \in S$ with $\phi \in s$.

Forcing with consistency properties

Definition

Let S be a consistency property for $\mathrm{L}_{\kappa\omega}$. Define the forcing notion $\mathbb{P}_S = (P, \leq)$:

- $P = S$,
- $s \leq t$ if and only if $t \subset s$.

Note that in this context the conditions in the definition of a consistency property talk about density. For example:

$$\forall v \phi(v) \in s, \ \textit{then } s \cup \{\phi(c)\} \in S \textit{ for any } c \in C,$$

is telling us that the sets of the form $D_{\phi(c)} = \{t \in S : \phi(c) \in t\}, c \in C$, are dense below the condition s.

Theorem (Model Existence Theorem, S. & Viale)

Let S be a consistency property for $\mathrm{L}_{\kappa\omega}$. If $s \in S$ and G is a V-generic filter containing s, then s is consistent in $V[G]$.

Corollary

A sentence $\phi \in \mathrm{L}_{\kappa\omega}$ has a model in some generic extension if and only if for some consistency propety S in V and some $s \in S$, $\phi \in s$.

Generalizing Tarski semantics

Instead of thinking about truth as a 1/0 function, lets take a more general perspective.

Let B be a boolean algebra, the truth value of a sentence in a B-valued model $\mathcal{M}$ is a function

$$\phi \mapsto [\phi]^{\mathcal{M}}_{\mathsf{B}} \in \mathsf{B}.$$

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$$\phi \mapsto [\phi]_{\mathsf{B}}^{\mathcal{M}} \in \mathsf{B}.$$

$$[\neg\phi]_{\mathsf{B}}^{\mathcal{M}} = \neg[\phi]_{\mathsf{B}}^{\mathcal{M}}$$

$$[\bigvee \Phi]_{\mathsf{B}}^{\mathcal{M}} = \bigvee_{\phi\in\Phi} [\phi]_{\mathsf{B}}^{\mathcal{M}} \qquad [\bigwedge \Phi]_{\mathsf{B}}^{\mathcal{M}} = \bigwedge_{\phi\in\Phi} [\phi]_{\mathsf{B}}^{\mathcal{M}}$$

$$[\exists v\phi(v)]_{\mathsf{B}}^{\mathcal{M}} = \bigvee_{m\in M} [\phi(m)]_{\mathsf{B}}^{\mathcal{M}} \qquad [\forall v\phi(v)]_{\mathsf{B}}^{\mathcal{M}} = \bigwedge_{m\in M} [\phi(m)]_{\mathsf{B}}^{\mathcal{M}}$$

Full boolean valued models

It might happen that $\mathcal{M}$ is a B-valued model realizing an existential sentence,

$$[\exists v \phi(v)]^{\mathcal{M}}_{\mathsf{B}} = 1_{\mathsf{B}},$$

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Definition

A B-valued model $\mathcal{M}$ is full if for every existential formula $\exists v \phi(v)$ there exists some $m \in M$ such that

$$[\exists v \phi(v)]^{\mathcal{M}}_{\mathsf{B}} = [\phi(m)]^{\mathcal{M}}_{\mathsf{B}}.$$

Model existence theorems

Theorem (Model Existence Theorem, Mansfield 1972)

Let S be a consistency property for $\mathrm{L}_{\kappa\kappa}$. For any $s \in S$ there exists $\mathcal{M}$ a boolean valued model realizing s,

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Theorem (Model Existence Theorem, Viale & S.)

Let S be a consistency property for $\mathrm{L}_{\kappa\omega}$. For any $s \in S$ there exists $\mathcal{M}$ a full boolean valued model realizing s,

$$[\bigwedge s]^{\mathcal{M}}_{\mathrm{B}} = 1_{\mathrm{B}}.$$

Completeness

Theorem (Boolean Completeness for $\mathrm{L}_{\kappa\kappa}$, Mansfield)

The following are equivalent for T, S sets of $\mathrm{L}_{\kappa\kappa}$-formulae.

1. $T \vDash_{\mathrm{BVM}} S$,
2. $T \vdash S$.

Theorem (Boolean Completeness for $\mathrm{L}_{\kappa\omega}$, S. & Viale)

The following are equivalent for T, S sets of $\mathrm{L}_{\kappa\omega}$-formulae.

1. $T \vDash_{\mathrm{Sh}} S$,
2. $T \vDash_{\mathrm{BVM}} S$,
3. $T \vdash S$.

Craig's interpolation

Theorem (Boolean Craig Interpolation for $\mathrm{L}_{\kappa\lambda}$)

Assume $\vDash_{\mathrm{BVM}} \phi \to \psi$ *with* $\phi, \psi \in \mathrm{L}_{\kappa\lambda}$. *Then there exists a sentence* θ *in* $\mathrm{L}_{\kappa\lambda}$ *such that*

- $\vDash_{\mathrm{BVM}} \phi \to \theta$, $\vDash_{\mathrm{BVM}} \theta \to \psi$,
- *all non logical symbols appearing in* θ *appear both in* ϕ *and* ψ.

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Omitting types theorems

Theorem (Boolean Omitting Types Theorem for $\mathrm{L}_{\kappa\lambda}$, S. & Viale)

Let T be a boolean satisfiable $\mathrm{L}_{\kappa\lambda}$-theory. Let $\Sigma(\overline{v})$ be an $\mathrm{L}_{\kappa\lambda}$ type and assume that it is not isolated by an $\mathrm{L}_{\kappa\lambda}$-sentence modulo T. Then there exists a boolean valued model $\mathcal{M}$ such that:

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Let T be a boolean satisfiable $\mathrm{L}_{\kappa\omega}$-theory. Let $\Sigma(v)$ be an $\mathrm{L}_{\kappa\omega}$-type and assume that it is not isolated by an $\mathrm{L}_{\kappa\omega}$-sentence modulo T. Then there exists a full boolean valued model $\mathcal{M}$ such that:

- $\mathcal{M} \vDash T$ *and* $\mathcal{M}$ *omits* Σ.