Dense metrizable subspaces in powers of Corson compacta

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"Dense metrizable subspaces in powers of Corson compacta"
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1. Some examples

Question. Which compact spaces have a dense metrizable subspace? Which compact spaces have a dense completely metrizable subspace?

Example 1.1

Let $X = [0, \omega_1]$ - a linearly ordered compact space. X is not metrizable because it does not have a countable base. By construction, X contains a dense subset of isolated points, therefore X has a dense completely metrizable subspace.

Example 1.2

Let $X = \beta\mathbb{N}$ - the Stone-Čech compactification of the naturals. Then X is not metrizable, and $\mathbb{N}$ is a dense completely metrizable subspace of X. The remainder $\beta\mathbb{N} \setminus \mathbb{N}$ is a compact space without any converging sequence, therefore without infinite metrizable subspaces.

Example 1.3

Let $X = [0,1]^{\mathfrak{c}}$ - Tychonoff cube or $X = \{0,1\}^{\mathfrak{c}}$ - Cantor cube of the weight continuum $\mathfrak{c}$. Then X is a separable compact, it contains large metrizable subspaces, but no metrizable subspace of X is dense in X. The reason: X does not have any G_δ point.

Claim 1.4

If X has a dense metrizable subset then X has a dense subset of G_δ points.

Example 1.5

Let X be the "double arrow" space. Formally, the underlying set of X is the linearly ordered set $[0,1] \times \{0,1\}$ with the lexicographic order. We equip X with the order topology.
X is a separable first-countable compact space, and every dense countable subset D of X is metrizable.
However X does not contain a dense *completely* metrizable subspace.

Definition 2.1

Let X be a compact Hausdorff space. Then

(i) X is called a *Corson compact* if X is homeomorphic to a subset of Σ-product of the real lines $\Sigma(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : \mathrm{supp}(x)$ is countable$\}$, where $\mathrm{supp}(x) = \{\gamma \in \Gamma : x_\gamma \neq 0\}$, for some set Γ.

(ii) X is called an *Eberlein compact* if X homeomorphically embeds into $(c_0(\Gamma), \tau_p)$, where $c_0(\Gamma) = \{x \in \mathbb{R}^\Gamma : \forall \epsilon > 0 \, \{\gamma \in \Gamma : |x(\gamma)| > \epsilon\}$ *is finite*$\}$.

(iii) X is called a *Talagrand compact* if $C_p(X)$ is a $\mathcal{K}$-analytic space.

(iv) X is called a *Gul'ko compact* if $C_p(X)$ is a Lindelöf Σ-space.

Denote by $\mathcal{E}$ - the class of all Eberlein compacts; $\mathcal{E}_1$ - the class of all Talagrand compacts; $\mathcal{E}_2$ - the class of all Gul'ko compacts; $\mathcal{K}$ - the class of all Corson compacts. It has been proven that

$$\mathcal{E} \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{K}$$

and all those classes are different.

Existence of dense metrizable subspaces in $X \in \mathcal{E}_2$

Theorem 2.2 (Namioka [1974])

Every Eberlein compact X has a dense G_δ metrizable subspace.

The following question arised naturally.

Problem (Talagrand [1979])

Does every compact space from the class $\mathcal{E}_1$, from the class $\mathcal{E}_2$ (i.e. Talagrand or Gul'ko compact) has a dense metrizable subspace?

Talagrand's problem has been resolved positively.

Theorem 2.3 (L. [1985])

Every Gul'ko compact X has a dense metrizable subspace.

Later but independently Gruenhage obtained a stronger statement.

Theorem 2.4 (Gruenhage [1987])

Every Gul'ko compact X has a dense G_δ metrizable subspace.

3. Corson compacts and dense metrizable subspaces

Initially, Corson compacts X which do not have a dense metrizable subspace were known assuming additional set-theoretic assumptions.

Let T be any ω_1- tree (all chains (= linearly ordered subsets) are at most countable). Denote by $\mathcal{A}$ the family of all chains $A \subset T$. Consider the following subset of the Cantor cube:

$$X_{\mathcal{A}} = \{\chi_A : A \in \mathcal{A}\} \subset \{0,1\}^T$$

Then $X_{\mathcal{A}}$ is a Corson compact.

Example 3.1

Let T be a Suslin tree. Then $X_{\mathcal{A}}$ is a Corson compact without dense metrizable subspace.

Example 3.2

Let $T = T(S)$ be the Todorcevic tree. Take any bistationary set $S \subset \omega_1$. $T(S)$ is the set of all countable subsets of S which are closed in ω_1. The order on $T(S)$ is defined as follows: $t_1 < t_2$ iff t_1 is a proper initial segment of t_2. Then T is a ω_1- tree. Claim: $X_{\mathcal{A}}$ is a Corson compact without dense metrizable subspaces.

Example 3.3

Let $S_1, S_2 \subset \omega_1$ be two bistationary sets such that $S_1 \cap S_2$ is non-stationary. Consider the tree $T = T(S_1) \otimes T(S_2)$. Here T as a set is $\{(t_1, t_2) : t_1 \in T_1, t_2 \in T_2,\ t_1 \cong t_2\}$. The order on $T(S_1) \otimes T(S_2)$ is defined as follows: $(s_1, s_2) < (t_1, t_2)$ iff $s_1 < t_1, s_2 < t_2$. Then $X_{\mathcal{A}_1}, X_{\mathcal{A}_2}$ are Corson compacts without dense metrizable subspaces, but their product $X_{\mathcal{A}_1} \times X_{\mathcal{A}_2}$ is a Corson compact which does have a dense metrizable subspace.

4. Countable power of Corson compacts and dense metrizable subspaces

Main Problem

Is there a Corson compact X such that its countable power X^ω

(a) does not have dense metrizable subspaces?

(b) does not have dense completely metrizable subspaces?

Example 3.1 continued

$X = X_{\mathcal{A}}$ is constructed on the Suslin tree. X^ω does have a dense metrizable subspace.

Example 3.2 continued

$X = X_{\mathcal{A}}$ is constructed on the Todorcevic tree. X^ω does have a dense metrizable subspace.

Example 4.1

There exists in ZFC a Corson compact X such that X^ω does not have a dense G_δ metrizable subspace, so the Main Problem (b) has a positive solution in ZFC.

Example 4.2

Assuming (CH) there exists a non-metrizable Corson compact X with a strictly positive probability Radon measure μ, i.e. $\mu(V) > 0$ for every nonempty open set $V \subset X$. Therefore X^ω also admits a strictly positive Radon measure. Hence $c(X^\omega) = \aleph_0$. It follows that X^ω cannot have a dense metrizable subspace, otherwise X^ω would be separable and then X would be metrizable. So, (CH) provides a consistent (positive) solution for the Main Problem 3(a).

Some topological cardinal invariants

(i) A family $\mathcal{P}$ of nonempty open subsets of X is called a π-base if for every nonempty open set $U \subset X$, there is $P \in \mathcal{P}$ such that $P \subset U$.

(ii) The minimal cardinality of a π-base for X is called the π-weight of X. Notation is $\pi w(X)$.

(iii) $c(X)$ denotes the cellularity of X, i.e. the supremum of the cardinalities of cellular families in X. The cardinal number $\widehat{c}(X)$ is defined as the minimal cardinal κ such that X does not have a cellular family of cardinality κ.

(iv) For every Corson compact $\pi w(X) = d(X) = w(X)$ holds.

(v) Always $c(X) \leq \widehat{c}(X)$ and $c(X) \leq \pi w(X)$.

Given any Corson compact X, we van find another Corson compact Y such that the product $X \times Y$ contains a dense metrizable subspace.

Proposition 4.3

Let X be any Corson compact. Denote by $A(\kappa)$ the one-point compactification of a discrete set of cardinality $\kappa = \pi w(X)$ and $Y = A(\kappa)^\omega$. Then the product $X \times Y$ contains a dense metrizable subspace.

The following result is a key tool.

Theorem 4.4

Let X be any Corson compact. Then X^ω has a dense metrizable subspace if and only if $\widehat{c}(X) > w(X)$.

Theorem 5.1

The following are equivalent

(1) There is a ccc Corson compact without a dense metrizable subspace.

(2) There is a compact ccc space which can be covered by ω_1 nowhere dense subsets.

(3) MA_{ω_1} fails.

Proof of (3) $\Rightarrow$ (1) in Theorem 5.1

If MA_{ω_1} fails, then there is a ccc poset $\mathbb{P}$ and an uncountable set $E \subset \mathbb{P}$ without an uncountable *centered* subset.
Recall that a subset C of $\mathbb{P}$ is called centered if for every finite $F \subset C$ there is $q \in \mathbb{P}$ such that $q \leq p$, for every $p \in F$.
Denote by $\mathcal{A}$ the family of all centered subsets $A \subset E$. Then $\mathcal{A}$ is an adequate family of sets and it consists only of at most countable sets. Consider the adequate Corson compact space:

$$X = X_{\mathcal{A}} = \{\chi_A : A \in \mathcal{A}\} \subset \{0,1\}^E$$

Then X contains the uncountable discrete set $\{\chi_{\{p\}} : p \in E\}$, therefore X is not metrizable. It can be shown that X is ccc. Further, X does not have a dense metrizable subspace, since otherwise the whole X would be metrizable.

Recall that a poset $(\mathbb{P}, \leq)$ is said to be *powerfully ccc* if each of its finite power is ccc. Powerfully ccc posets have turned out to be sufficient for all known applications of Martin's Axiom, and it is still unknown whether Martin's Axiom is equivalent to Martin's Axiom restricted to powerfully ccc posets.
If MA_{ω_1} fails, then there is a powerfully ccc poset $\mathbb{P}$ and an uncountable set $E \subset \mathbb{P}$ without an uncountable *centered* subset. Similarly to the previous Theorem 5.1 we prove

Theorem 5.2

The following are equivalent

(1) There is a Corson compact X such that X^ω is ccc but it does not have a dense metrizable subspace.

(2) There is a compact space X such that X^ω is ccc and can be covered by ω_1 nowhere dense subsets.

(3) MA_{ω_1} for powerfully ccc posets fails.

(4) There is a ccc Corson compact X of weight ω_1 such that X^ω does not have a dense metrizable subspace.

Corollary 5.3

Assume that a Suslin tree exists. Then there is a Corson compact X such that X^ω does not have a dense metrizable subspace.

Problem 5.4

Is there a natural weakening of Martin's Axiom which is equivalent to the statement that X^ω has a dense metrizable subspace for every ccc Corson compact X?

The most natural class of Corson compacta X such that X^ω is ccc is the class of Corson compacts which support a strictly positive Radon probability measure. We give a new combinatorial proof of the following result.

Theorem 5.5

(Kunen and van Mill) The following are equivalent

(1) Every Corson compact with a strictly positive Radon probability measure is metrizable.

(2) MA_{ω_1} for measure algebras.

Recall that a Suslin subtree T of $\omega^{<\omega_1}$ is called *coherent* if for every
$s, t \in T, \{\beta \in dom(s) \cap dom(t) : s(\beta) \neq t(\beta)\}$ is finite.

Theorem 5.6

Assume that a coherent Suslin tree exists. Then there is a perfectly normal Corson compact X such that X does not have a dense metrizable subspace, but the square X^2 does have a dense metrizable subspace.

Theorem 5.7

Assume that a full Suslin tree exists. Then there is a perfectly normal Corson compact X such that X^n does not have a dense metrizable subspace, for every finite n, but the countable power X^ω does have a dense metrizable subspace.

Problem 5.8

Is there a ZFC example of a Corson compact X such that X does not have a dense metrizable subspace, but the square X^2 does have a dense metrizable subspace?

6. $c(X^\omega)$ versus density $d(X)$

Theorem 6.1

Let $\kappa = \aleph_\omega$. Consistently, there is a Corson compact X such that $c(X^\omega) = \kappa < d(X)$.

Problem 6.2

Is there a ZFC example of a Corson compact X such that $c(X^\omega) < d(X)$?

If we are willing to relax a little bit the requirement that X is a Corson compact space, we can obtain an example in ZFC.
A compact space X is called a τ-*Corson compact* if X can be embedded into
$\Sigma_\tau(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : |\operatorname{supp}(x)| \leq \tau\}$.
Thus, an $\aleph_0$-Corson compact is simply a Corson compact.

Theorem 6.3

In ZFC there is an $\aleph_3$-Corson compact X such that $c(X^\omega) < d(X)$.

Thank you!