A dichotomy for countable unions of smooth Borel equivalence relations

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- We say that E continuously embeds into F, denoted by $E \sqsubseteq_c F$, if there is a continuous embedding from E to F.

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which is exhaustive in the sense that every Borel equivalence relation is either bireducible with one of the elements of this initial segment, or is strictly greater than $\mathbb{E}_0$.

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Proposition (Folklore)

The relation $\mathbb{E}_1$ is not essentially countable.

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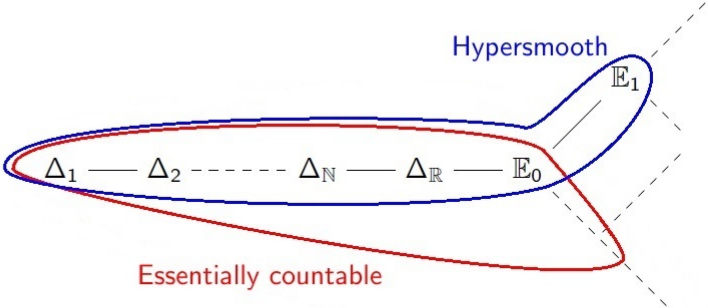

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There are other examples, for instance the disjoint union of $\mathbb{E}_1$ and of a non-hypersmooth countable Borel equivalence relation.

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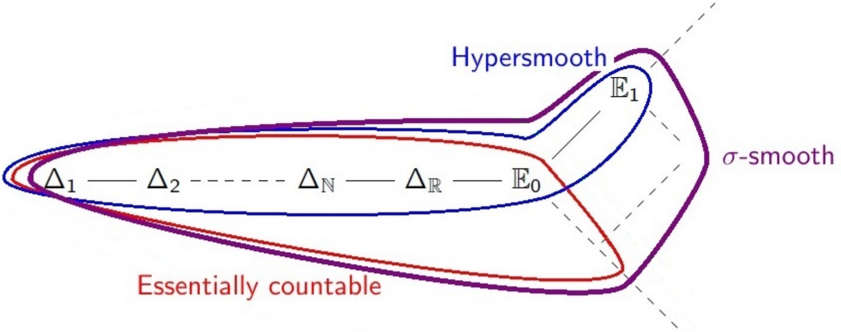

Idealisticity

A Borel equivalence relation E on a Polish space X is said to be idealistic if there is an E-invariant assignment $x \mapsto \mathcal{I}_x$ mapping each point in X to a σ-ideal on X in such a way that:

- $\forall x \in X,\ [x]_E \notin \mathcal{I}_x$;
- for every $x \in X$ and every uncountable family $(B_i)_{i \in I}$ of pairwise disjoint Borel subsets of X, one of the B_i's is in $\mathcal{I}_x$
- For every Polish space Y and every Borel set $R \subseteq X \times Y \times X$, the set $\{(x, y) \in X \times Y \mid R_{x,y} \in \mathcal{I}_x\}$ is Borel.

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Conjecture ((Almost) Kechris–Louveau)

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Kechris–Louveau's dichotomy solves this conjecture in the special case of hypersmooth Borel equivalence relations. Our dichotomy solves it in the special case of σ-smooth Borel equivalence relations.

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Denote by $\mathcal{F}^{\leqslant_B}$ the family of all equivalence relations on Polish spaces that are Borel reducible to an element of $\mathcal{F}$, and by $\sigma(\mathcal{F})$ the class of all equivalence relations on Polish spaces that can be expressed as countable unions of subequivalence relations belonging to $\mathcal{F}$.

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If we take for $\mathcal{F}$ the class of all idealistic potentially F_σ equivalence relations on Polish spaces, then we get that the Kechris–Louveau conjecture is satisfied by all equivalence relations that can be expressed as a countable union of subequivalence relations that are Borel reducible to idealistic potentially F_σ equivalence relations on Polish spaces.

Thank you for your attention!