Indestructibility of the tree property over models of PFA

Šárka Stejskalová

Department of Logic, Charles University
Institute of Mathematics, Czech Academy of Sciences

logika.ff.cuni.cz/sarka

January 30, 2022

The tree property

Recall the following definition:

Definition

Let κ be a regular cardinal. A κ-tree is called *Aronszajn* if it has no cofinal branch. The tree property holds at κ, TP(κ), if there are no κ-Aronszajn trees.

The tree property is a compactness property which has been recently extensively studied. It is known that together with inaccessibility of κ it characterizes weak compactness. It is open whether the tree property can hold on all regular cardinals $> \omega_1$, but there are methods how to get the tree property at infinitely many successive cardinals.

The tree property

Some basic properties:

- TP(ω) and $\neg$TP(ω_1).
- (Specker) If $\kappa^{<\kappa} = \kappa$ then there exists a κ^+-Aronszajn tree. Therefore $\neg$TP(κ^+).
  - If GCH then $\neg$TP(κ^{++}) for all $\kappa \geq \omega$.
  - TP(κ^{++}) then $2^\kappa > \kappa^+$.
- PFA implies TP(ω_2).

Guessing models

Definition

Let θ be a cardinal, and let $M \prec H(\theta)$ and $z \in M$.

1. A set $d \subseteq z$ is *M-approximated* if $d \cap a \in M$ for all countable $a \in M$.
2. A set $d \subseteq z$ is *M-guessed* if there is an $e \in M$ such that $d \cap M = e \cap M$.
3. M is a *guessing model* if for every $z \in M$, if $d \subseteq z$ is M-approximated, it is M-guessed.

Guessing model principle

Definition

We say that the *Guessing model principle* holds at ω_2, and write $\mathrm{GMP}(\omega_2)$, if the set

$$\{M \prec H(\theta) \,|\, |M| < \omega_2 \text{ and } M \text{ is a guessing model}\}$$

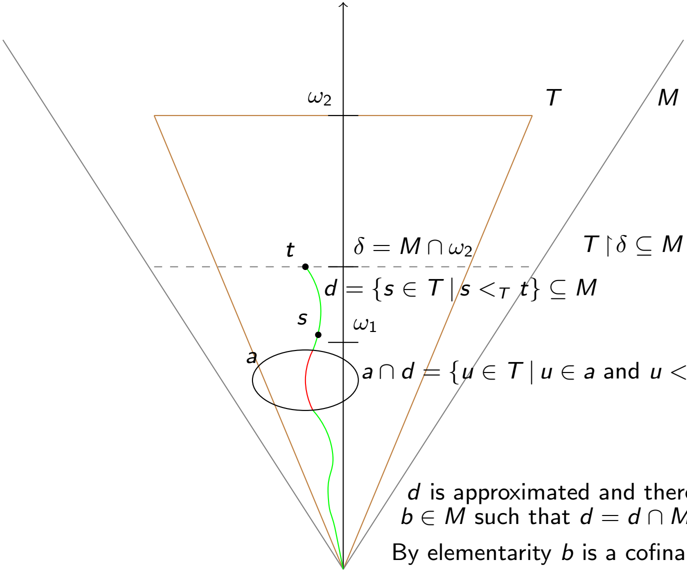
is stationary in $\mathcal{P}_{\omega_2} H(\theta)$ for every $\theta \geq \omega_2$

Viale and Weiss in [4] proved following: PFA implies $\mathrm{GMP}(\omega_2)$.

Note that $\mathrm{GMP}(\omega_2)$ implies the tree property at ω_2.

GMP(ω_2) implies TP(ω_2)

$M \prec H(\omega_3)$ is a guessing model such that $T \in M$, $|M| = \omega_1$ and $\omega_1 \subseteq M$



d is approximated and therefore there is $b \in M$ such that $d = d \cap M = b \cap M$.

By elementarity b is a cofinal branch in T

Indestructibility of the tree property

Theorem (Honzik, S., [1])

GMP(ω_2), *and hence* PFA, *implies that the tree property at* ω_2 *is indestructible under the single Cohen forcing at* ω, *i.e. if* V *is a transitive model satisfying* GMP(ω_2) *and* G *is* Add($\omega, 1$)*-generic over* V, *then* $V[G]$ *satisfies the tree property at* ω_2.

Remark: It seems to be a hard question in general to determine whether a small forcing such as the Cohen forcing Add($\omega, 1$) can add a large Aronszajn tree. It can be shown for many specific models that it cannot, but in general the question is wide open (no counter-example is known so far). Our results says that no counter-example can be found for models of PFA.

Derived systems

In order to show that an ω_2-Aronszajn trees cannot exist in a generic extension $V[\mathrm{Add}(\omega, 1)]$, we will work in the ground model and work with a system derived from an $\mathrm{Add}(\omega, 1)$-name for an ω_2-tree.

Well-behaved strong (ω_1, ω_2)-systems

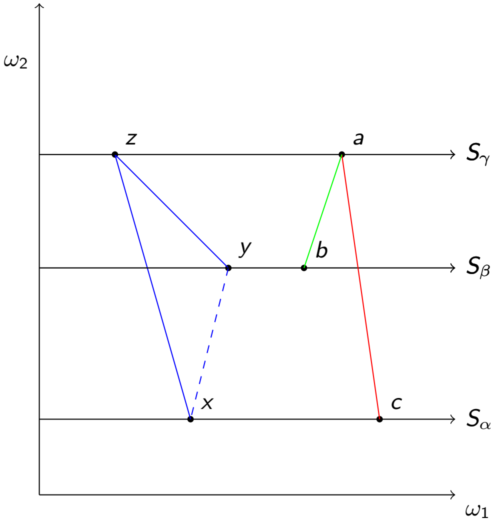
Let $D \subseteq \omega_2$ be unbounded in ω_2. For each $\alpha \in D$, let $S_\alpha \subseteq \{\alpha\} \times \omega_1$ and let $S = \bigcup_{\alpha \in D} S_\alpha$. Moreover, let I be an index set of cardinality at most ω and $\mathcal{R} = \{<_i \mid i \in I\}$ a collection of binary relations on S. We say that $\langle S, \mathcal{R} \rangle$ is an (ω_1, ω_2)-*system* if the following hold for some D:

1. For each $i \in I$, $<_i$ is irreflexive and transitive.
2. For each $i \in I$, $\alpha, \beta \in D$ and $\gamma, \delta < \omega_1$; if $(\alpha, \gamma) <_i (\beta, \delta)$ then $\alpha < \beta$.
3. For each $i \in I$, and $\alpha < \beta < \gamma$, $x \in S_\alpha$, $y \in S_\beta$ and $z \in S_\gamma$, if $x <_i z$ and $y <_i z$, then $x <_i y$.
4. For all $\alpha < \beta$ there are $y \in S_\beta$ and $x \in S_\alpha$ and $i \in I$ such that $x <_i y$.

We call a (ω_1, ω_2)-system $\langle S, \mathcal{R} \rangle$ a *strong* (ω_1, ω_2)-*system* if the following strengthening of item (iv) holds:

4' For all $\alpha < \beta$ and for every $y \in S_\beta$ there are $x \in S_\alpha$ and $i \in I$ such that $x <_i y$.

Well-behaved strong (ω_1, ω_2)-systems



Cofinal branches through system

A *branch* of the system is a subset B of S such that for some $i \in I$, and for all $a \neq b \in B$, $a <_i b$ or $b <_i a$. A branch B is *cofinal* if for each $\alpha < \omega_2$ there are $\beta \geq \alpha$ and $b \in B$ on level β.

Systems derived from Cohen forcing

Let $\mathrm{Add}(\omega, 1)$ forces that $\dot{T}$ is an ω_2-tree. The derived system has domain $\omega_1 \times \omega_2$, and is equipped with binary relations $<_p$ for $p \in \mathrm{Add}(\omega, 1)$, where

$$x <_p y \Leftrightarrow p \Vdash x \mathrel{\dot{<}_T} y. \tag{1}$$

Indestructibility of the tree property

Recall that we want to prove that the tree property at ω_2 is indestructible by Cohen forcing $\mathrm{Add}(\omega, 1)$ over transitive models of $\mathrm{GMP}(\omega_2)$.

- Assume for contradiction that $\dot{T}$ is forced by the weakest condition in $\mathrm{Add}(\omega, 1)$ to be an ω_2-tree.
- Let $S(\dot{T})$ be the derived system with respect to $\dot{T}$
- Similarly to the proof that $\mathrm{GMP}(\omega_2)$ implies that every ω_2-tree has a cofinal branch, one can show that under $\mathrm{GMP}(\omega_2)$ every well-behaved strong (ω_1, ω_2)-system has a cofinal branch; i.e. there is a cofinal branch B in $S(\dot{T})$ with respect to $<_p$ for some $p \in \mathrm{Add}(\omega, 1)$
- If G be a $\mathrm{Add}(\omega, 1)$-generic containing p. Then B is a cofinal branch in $\dot{T}^G$.

σ-centered forcing

- Note that derived systems can be naturally used with σ-centered forcings: Let $\mathbb{P}$ be a σ-centered forcing. Let us write $\mathbb{P} = \bigcup_{n<\omega} \mathbb{P}_n$. Then we can define the relations $<_n$ for $n < \omega$, where

$$x <_n y \Leftrightarrow (\exists p \in \mathbb{P}_n)\, p \Vdash x \mathrel{\dot{<}_T} y. \tag{2}$$

- Let us say a few words about obstacles to generalising the argument to σ-centered forcings: Suppose $S(\dot{T})$ is the derived system with respect to some σ-centered forcing $\mathbb{P}$.
- Arguing as we did, one can show that there is a cofinal branch B in $S(\dot{T})$ with respect to some $<_n$, $n < \omega$. However the proof that B is a cofinal branch in the generic extension may fail because if G is $\mathbb{P}$-generic, then it may be false that for all (or sufficiently many) x, y in B there is some $p \in G \cap \mathbb{P}_n$ forcing $x <_{\dot{T}} y$.

Theorem (Honzik, S., [1])

GMP(ω_2), *and hence also* PFA, *implies that the negation of the weak Kurepa Hypothesis is indestructible under any* σ*-centered forcing, i.e. if* V *is a transitive model satisfying* GMP(ω_2), $\mathbb{P}$ *is* σ*-centered, and* G *is* $\mathbb{P}$*-generic over* V, *then* $V[G]$ *satisfies the negation of the weak Kurepa Hypothesis at* ω_1*.*

O ver specific generic extensions, indestructibility is easier to verify. Let us review some results for the Mitchell extension (forcing with $\mathbb{M}(\omega,\kappa)$, where κ is weakly compact):

- (Todorcevic, [3]) Todorcevic showed in [3] that the tree property at ω_2 in the Mitchell model $V[\mathbb{M}(\omega,\kappa)]$ is indestructible under any finite-support iteration of ccc forcing notions which have size less than ω_2 and do not add a new cofinal branch to ω_1-trees.
- (Honzik, S., [2]) The tree property at ω_2 in $V[\mathbb{M}(\omega,\kappa)]$ is indestructible under all ccc forcings which live in $V[\mathrm{Add}(\omega,\kappa)]$.

Some open questions:

1. Is the tree property at ω_2 indestructible under all σ-centered forcings over every model which satisfies GMP(ω_2) or PFA?
2. Or more modestly, is the tree property at ω_2 indestructible under Add(ω, ω_1) under the same assumptions?
3. Can our result about the Mitchell model be extended to all ccc forcing notions in $V[\mathbb{M}(\omega, \kappa)]$? Or more generally, is there a model V^* over which TP(ω_2) is indestructible under all ccc forcings?
4. More specifically, neither our result nor Todorcevic's result applies to an iteration of ω_1-Suslin trees of length ω_2. Can either of these results be extended to this forcing?

- Radek Honzik and Šárka Stejskalová, *Indestructibility of some compactness principles over models of* PFA *and* GMP, submitted.
- ______, *Indestructibility of the tree property*, The Journal of Symbolic Logic **85** (2020), no. 1, 467–485.
- Stevo Todorcevic, *Some consequences of* $\mathsf{MA} + \neg\mathsf{wKH}$, Topology and its Applications **12** (1981), no. 2, 187–202.
- Matteo Viale and Christoph Weiss, *On the consistency strength of the proper forcing axiom*, Advances in Mathematics **228** (2011), 2672–2687.