# Limits of weighted hyperfinite graphs

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# Schreier graphs

Fix a f. g. group $\Gamma$ and its finite system of generators $S = S^{-1}$.

**Goal:** What does it mean that a sequence of Schreier graphs $(G_n)_{n=1}^{\infty}$ converges?

Rooted Schreier graph $(G, v)$: Schreier graph $G$ with a distinguished vertex $v$.
$\mathcal{S}_\Gamma$: space of (the isomorphism classes of) finite connected rooted Schreier graphs of $\Gamma$ (w.r.t. $S$).
Two rooted graphs on $\mathcal{S}_\Gamma$ are close if large balls around their roots are isomorphic. This topology makes $\mathcal{S}_\Gamma$ into a space homeomorphic with the Cantor set.

# Local convergence

Given a finite Schreier graph $G$ we get a probability measure $\mu_G$ on $\mathcal{S}_\Gamma$ by taking

$$\mu_G = \frac{1}{|G|} \sum_{v \in G} \delta_{(C_v, v)},$$

where $C_v$ is the component of $v$.

Then a sequence $(G_n)_{n=1}^{\infty}$ is said to be convergent if $\mu_{G_n}$ weakly converge to some measure $\mu$. [Benjamini-Schramm, 2001]

# Graphings

Given an action $\alpha : \Gamma \curvearrowright (X, \mu)$, where $(X, \mu)$ is a standard probability space, there is a measurable map

$$M : X \to \mathcal{S}_\gamma,$$

given by

$$M(x) := (G_x, x),$$

where $G_x$ is the Schreier graph of $\alpha$ on the orbit of $x$.

Then, the graphing $\mathcal{G}_\alpha$ of the action $\alpha$ is called the limit of a sequence $(G_n)_{n=1}^{\infty}$ if $\mu_{G_n}$ weakly converge to the pushforward $M_*\mu$.

# Graphings II

**Theorem**

*If $\mu \in Prob(\mathcal{S}_\Gamma)$ is a limit of finite graphs, then there exists a p.m.p. action $\alpha : \Gamma \curvearrowright (X, \nu)$ such that $\mu = M_*(\nu)$.*

**Question**

Is the graphing of any p.m.p. action of a finitely generated group a limit of finite Schreier graphs? [Aldous-Lyons Conjecture, '07]

# Example

Consider $\Gamma = \mathbb{Z}$ acting on the set $[n]$ by addition modulo $n$.
Taking the generator system $S = \{\pm 1\}$, we obtain that the corresponding Schreier graph is an $n$-cycle; call it $G_n$.

Then, $\mu_n$ converges to $\mu$ which is the measure concentrated on a two-ended line graph (with root anywhere). Therefore, the graphing of any free p.m.p. action of $\mathbb{Z}$ represents a limit of $(G_n)_{n=1}^{\infty}$.

# Hyperfiniteness I

## Definition

A graphing $\mathcal{G}$ on a probability space $(X, \mu)$ is called *hyperfinite* (or $\mu$-hyperfinite) if for any $\epsilon > 0$ and $K \in \mathbb{N}$ there is a set $Z \subseteq X$ such that the components of $\mathcal{G} - Z$ have at most $K$ elements.

## Definition

A sequence of graphs $(G_n)_{n=1}^{\infty}$ is called *hyperfinite* if for any $\epsilon > 0$ and $K \in \mathbb{N}$ there are sets of nodes $Z_n \subseteq V(G_n)$ such that all components of $G_n - Z_n$ have at most $K$ nodes and $|Z_n| < \epsilon |V_n|$.

## Theorem (Schramm, '08)

*Let $\mathcal{G}$ be a graphing which is a local limit of a graph sequence $(G_n)_{n=1}^{\infty}$. Then $\mathcal{G}$ is hyperfinite if and only if $(G_n)_{n=1}^{\infty}$ is hyperfinite.*

# Weighted setting

We wish to do a similar thing for graphs with weights: $(G, w)$, where $G$ is finite and $w$ is a probability measure on $V(G)$ such that $w(v) > 0$ for all $v \in V(G)$.

**Issue:** We need to somehow preserve the information about the measure $w$ on the graph.
Thus, we define a *cocycle function* on the (directed) edges of $G$:

$$\rho(x, y) := \frac{w(y)}{w(x)}.$$

# Rooted cocycles

The name ‘cocycle’ comes from the fact that for any directed circuit $v_0, v_1, \ldots, v_n = v_0$ the following holds:

$$\prod_{k=0}^{n-1} \rho(v_k, v_{k+1}) = 1.$$

In particular, for any edge $(v, w)$, $\rho(v, w) = \rho(w, v)^{-1}$.

$C = (G, \rho)$ is called simply a cocycle.
We let $\mathfrak{C}$ denote the set of finite connected rooted cocycles.
$(r, \epsilon)$-neighborhood of a cocycle $C$ is the set of all rooted cocycles $D$ whose $r$-ball around the root is isomorphic to the $r$-ball in $C$ and on each respective edge the cocycle functions differ by no more than $\epsilon$.

# Weighted local convergence

Similarly as before, any finite weighted graph $C = (G, w)$ with an associated cocycle function $\rho$ induces a measure on $\mathfrak{C}$ by

$$\mu_C := \sum_{v \in V(G)} w(v) \delta_{(C_v, v, \rho)}.$$

A sequence of weighted graphs $(G_n, w_n)_{n=1}^{\infty}$ converges if the corresponding cocycle functions $\rho_n$ are bounded by some $K > 0$ and the measures $\mu_{C_n}$ weakly converge.

# Nonsingular actions

A limit measure of a sequence of finite weighted graphs can also be represented with a "graphing" of an action of $\Gamma$.
An action $\alpha : \Gamma \curvearrowright (X, \nu)$ is called nonsingular if it preserves the measure class, i.e. for any $\gamma \in \Gamma$ and any measurable $A \subseteq X$

$$\mu(A) = 0 \quad \text{iff} \quad \mu(\alpha_\gamma A) = 0.$$

**Note:** For nonsingular actions, for each $\gamma \in \Gamma$, the Radon-Nikodým derivative of the map $\alpha_\gamma$ exists on a co-null subset of $X$. Moreover, the Radon-Nikodým derivatives form a cocycle on $X$.

# Representation of a limit measure II

Therefore, there exists a map $M : (X, \nu) \to \mathfrak{C}$ which assigns to each $x \in X$ the Schreier graph on the orbit of $x$ together with the cocycle function arising from the Radon-Nikodým derivatives of the maps $\alpha_s, s \in S$.

When $M_* \nu$ is a weak limit of the measures $\mu_{C_n}$, we say that the nonsingular graphing of $\alpha$ is a limit of the of sequence weighted graphs $(G_n, w_n)_{n=1}^{\infty}$.

# Hyperfiniteness II

## Definition

A graphing $\mathcal{G}$ of a nonsingular action $\alpha$ on a probability space $(X, \mu)$ is called *hyperfinite* if for any $\epsilon > 0$ and $K \in \mathbb{N}$ there is a set $Z \subseteq X$ such that the components of $\mathcal{G} - Z$ have at most $K$ elements.

## Definition

A sequence of weighted graphs $(G_n, w_n)_{n=1}^{\infty}$ is called *hyperfinite* if for any $\epsilon > 0$ and $K \in \mathbb{N}$ there are sets of nodes $Z_n \subseteq V(G_n)$ such that all components of $G_n - Z_n$ have at most $K$ nodes and $\sum_{v \in Z_n} w_n(v) < \epsilon$.

## Theorem (Elek, K)

*Let $\mathcal{G}$ be a graphing of a non-singular action which is a local limit of a weighted graph sequence $(G_n, w_n)_{n=1}^{\infty}$. Then $\mathcal{G}$ is hyperfinite if and only if $(G_n, w_n)_{n=1}^{\infty}$ is hyperfinite.*