A characterization of metrizability through games

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Given a set X, a metric is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq d(x, y) + d(y, z)$.

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Every metric space generates a topology with base

$$\mathcal{B} = \{B_d(x, \varepsilon) \mid \varepsilon \in \mathbb{R}^+\},$$

where

$$B_d(x, \varepsilon) = \{y \in X \mid d(x, y) <_{\mathbb{G}} \varepsilon\}.$$

A topological space (X, τ) is said metrizable if there is a metric d on X generating τ.

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Given a set X and a structure $\mathbb{G} = \langle G, +_{\mathbb{G}}, 0_{\mathbb{G}}, \leq_{\mathbb{G}} \rangle$, a $\mathbb{G}$-metric is a function $d \colon X^2 \to \mathbb{G}_{\geq 0_{\mathbb{G}}}$ such that:

1. $d(x, y) = 0_{\mathbb{G}}$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq_{\mathbb{G}} d(x, y) +_{\mathbb{G}} d(y, z)$.

A topological space (X, τ) is said $\mathbb{G}$-metrizable if there is a $\mathbb{G}$-metric d on X generating τ.

How different/more general are $\mathbb{G}$-metrics?

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$\mathrm{Deg}(\mathbb{G})$ is the coinitiality of $\mathbb{G}^+ = \{\varepsilon \in \mathbb{G} \mid \varepsilon >_{\mathbb{G}} 0_{\mathbb{G}}\}$.

Remark: If X is $\mathbb{G}$-metrizable for $\mathrm{Deg}(\mathbb{G}) = \mu$, then the smallest size of a local base at any non-isolated point is μ.

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A space is said μ-**metrizable** if it is $\mathbb{G}$-metrizable for some *totally ordered group*[1] $\mathbb{G}$ with deg($\mathbb{G}$) $= \mu$.

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The countable case: ω-metrizability

Fact 1: metrizable $\Leftrightarrow$ ω-metrizable.

Fact 2: metrizable $\not\Rightarrow$ $\mathbb{G}$-metrizable for all $\mathbb{G}$ (with $\mathrm{Deg}(\mathbb{G}) = \mu$).

For example, $\mathbb{R}$ is not $\mathbb{Q}$-metrizable.

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A space is **ultrametrizable** if it is $\mathbb{G}$-metrizable for $\mathbb{G} = \langle \mathbb{R}, \max, 0, \leq \rangle$.

Fact 3: ultrametrizable $\Leftrightarrow$ $\mathbb{G}$-metrizable for all $\mathbb{G}$ with $\mathrm{Deg}(\mathbb{G}) = \mu$.

The uncountable case: μ-metrizability

Let $\mathbb{G}$ range among totally ordered (continuous semi)groups of degree μ.

Fact: If $\mu > \omega$, the following are equivalent for a space X:

- X is $\mathbb{G}$-metrizable for some $\mathbb{G}$ (μ-metrizable);
- X is $\mathbb{G}$-metrizable for every $\mathbb{G}$ (μ-ultrametrizable).

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Example: ${}^{\mu}\lambda$ is μ-metrizable.

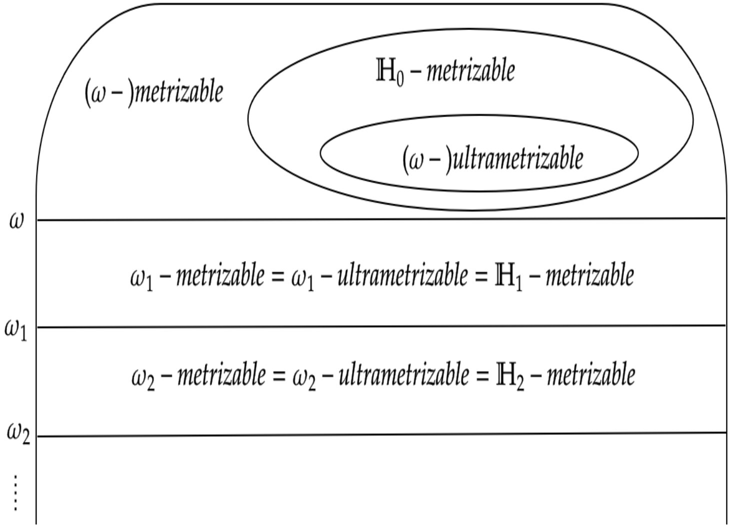

Figure 1: Classification of $\mathbb{H}$-metrizable non-discrete spaces for some structure $\mathbb{H}$. $\mathbb{H}_0$, $\mathbb{H}_1$ and $\mathbb{H}_2$ are totally ordered (continuous semi)groups with $\mathrm{Deg}(\mathbb{H}_i) = \omega_i$.

Characterizations.

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Second countable spaces: Urysohn and Sikorski

The **weight** $w(X)$ is the smallest size of a base for the topology.

Theorem (Urysohn Metrization Theorem)

Suppose $w(X) = \omega$. Then, X is metrizable if and only if X is T_3.

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X is μ-additive if intersections of $< \mu$-many opens are open.
Every μ-metrizable space is μ-additive.

Theorem (Sikorski 1950)

Suppose $w(X) = \mu$. Then, X is μ-metrizable if and only if X is T_3 and μ-additive.

*(**Remark:** Every space is ω-additive.)*

Characterizing metrizability: Bing, Nagata, Smirnov, Arhangel'skij, ...

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A family $\mathcal{A}$ of open subsets of topological space X is said **locally finite** (resp., locally $< \gamma$-small) if every point has an open neighborhood that intersect only finitely many (resp., $< \gamma$) sets from $\mathcal{A}$.

A $\mathsf{NS}_\delta^\gamma$-base is a base that is the union of δ-many locally $< \gamma$ families.

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Theorem (Bing-Nagata-Smirnov Metrization Theorem)

The following are equivalent:

1. *X is metrizable*
2. *X is T_3 with a $\mathrm{NS}_\omega^\omega$-base.*
3. *X is T_3 with a NS_ω^2-base.*

(**Remark:** $w(X) = \omega$ implies NS_ω^2-base).

A base $\mathcal{B}$ is **regular** if for every open U and for every $x \in U$ there is an open set V such that $x \in V \subseteq U$ and only finitely many elements of $\mathcal{B}$ meets both V and $X \setminus U$.

Remark: If $\mathcal{B}$ is regular, then:

1. $(\mathcal{B}, \supseteq)$ is wellfounded.
2. $\mathrm{ht}(\mathcal{B}, \supseteq) = \omega$ (every $B \in \mathcal{B}$ has finitely many predecessors).
3. Every level $\mathrm{Lev}_\alpha(\mathcal{B}, \supseteq)$ is locally finite.

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Theorem (Arhangel'skij metrization Theorem)

X is metrizable if and only if it is T_1 and has a regular base.

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Remark: Every tree base of height ω is a regular base.

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1. *X is ultrametrizable;*
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3. *X has a $\mathrm{NS}^{\omega}_{\omega}$-base of clopens;*
4. *X has a base union of ω-many clopen partitions;*

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4. *X has a base union of ω-many clopen partitions;*
5. *X has a tree base of height ω;*
6. *X is homeomorphic to a subset of ${}^{\omega}\lambda$ for some λ;*

Metrizability ***VS*** *ultrametrizability:*

Metrizable	Ultrametrizable
	Metrizable and Lebesgue zero-dimensional
$\mathrm{NS}^{\omega}_{\omega}$-base	$\mathrm{NS}^{\omega}_{\omega}$-base of clopens
NS^{2}_{ω}-base	Base union of ω-many clopen partitions
Regular base	Tree base of height ω
	$\cong A \subseteq {}^{\omega}\lambda$ for some λ

Characterizing μ-metrizability and μ-ultrametrizability for μ uncountable:
Artico, Hodel, Moresco, Nyikos, Reichel, Shu-Tang, Sikorski, ...

Recall: If $\mu > \omega$, every μ-metrizable space is Lebesgue zero-dimensional.

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Recall: If $\mu > \omega$, every μ-metrizable space is Lebesgue zero-dimensional.

Theorem (Various authors)

If $\mu > \omega$, the following are equivalent:

- *μ-metrizable.*
- *μ-metrizable and Lebesgue zero-dimensional.*
- *μ-ultrametrizable.*
- *X is μ-additive and has a NS_μ^δ-base (for some/every $2 \leq \delta \leq \mu$).*
- *X is μ-additive and has a NS_μ^δ-base of clopens (for $2 \leq \delta \leq \mu$).*
- *X is μ-additive and has a μ-regular base.*
- *X is μ-additive and has a tree base of height μ.*
- *X is homeomorphic to a subset of ${}^{\mu}\lambda$ (with bounded top.) for some λ.*

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So what grants (μ-)metrizability?

Answer: Basically, two things:

1. X has some paracompactness property: we can refine covers into (unions of) locally finite covers.
2. The space has "countable height" (or height μ for μ-metrizability).

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Conjecture: *X is μ-metrizable if and only if it is paracompact and every point has a local base of size μ?* (**No**: Sorgenfrey line)

The μ-**uniform local base game**: at every round $\alpha < \mu$, player I pick a point $x_\alpha \in X$, and player II replies with an open set V_α containing x_α.

$$\begin{array}{c|cccccccccc} \textbf{I} & x_0 & & x_1 & & \dots & x_\gamma & & \dots \\ \hline \textbf{II} & & V_0 & & V_1 & \dots & & V_\gamma & \dots \end{array}$$

At the end of the game, player II wins if $\bigcap_{\alpha<\mu} V_\alpha = \varnothing$ or if $\{V_\alpha \mid \alpha < \mu\}$ is a local base of a point $x \in X$, otherwise I wins.

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II		V_0		V_1	$\ldots$		V_γ		$\ldots$

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Definition

We say that a topological space is μ-**uniformly based** if player II has a winning strategy in the μ-uniform local base game.

Remark: Every point of a μ-unfiromly based space has a local base of size at most μ.

Remark: In every μ-metrizable space, II has a winning tactic in the μ-uniform local base game: let her play spheres of vanishing radii.

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Theorem (A., Motto Ros)

X is μ-metrizable if and only if it is μ-additive, paracompact and μ-uniformly based.

Corollary (A., Motto Ros)

X is metrizable if and only if it is paracompact and ω-uniformly based.

NS-bases ***VS*** *tree bases* ***VS*** *μ-ULB*

Recall: If X is μ-additive (and T_3), then TFAE:

1. Tree base of height μ;
2. $\mathsf{NS}^{\delta}_{\mu}$-base (for any δ);
3. μ-ULB and paracompact.

NS-bases ***VS*** *tree bases* ***VS*** *μ-ULB*

Recall: If X is μ-additive (and T_3), then TFAE:

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Proposition (A., Motto Ros)

Suppose $\mu > \omega$ and and the μ-Borel hierarchy does not collapse before Σ^0_2 on 2^γ for some $\gamma < \mu$.
There exists a T_3, (Lebesgue zero-dimensional, paracompact,) μ-uniformly based space X with a tree base of height μ that is not NS^{μ}_{μ}.

Proposition (A., Motto Ros)

There exists a μ-additive space X with a tree base where every point has a local base of size μ, but X is not μ-metrizable (nor NS_μ^μ nor μ-ULB).

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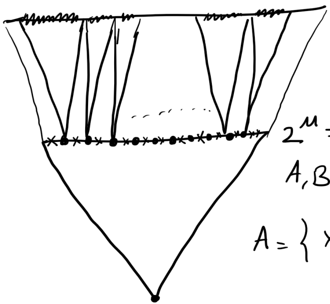

$X = \{x \in 2^{\mu+\mu} \mid x \upharpoonright \mu \in A\} \subseteq 2^{\mu+\mu}$

$2^\mu = A \cup B$

A, B DENSE IN 2^μ

$A = \{x \in 2^\mu \mid x(\alpha) = 0$ FOR $<\mu$-MANY $\alpha\}$

Proposition (A., Motto Ros)

Every space with a tree base of height μ is μ-uniformly based (but II does not have necessarily a tactic).

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Proposition (A., Motto Ros)

Every NS_μ^μ-space is μ-uniformly based.

Proposition (A., Motto Ros)

In every δ-additive NS_μ^δ-space, player II has a winning tactic in the μ-uniform local base game.

X is (δ, μ)-paracompact if every open cover of X can be refined into a cover that is the union of μ-many locally $< \delta$-small open cover.

Theorem (A., Motto Ros)

Suppose X is δ-additive. Then, X has a $\mathsf{NS}_{\mu}^{\delta}$-base if and only if it is (δ, μ)-paracompact and player II has a winning tactic in the μ-uniform local base game.

Thank you for the attention!

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