Free group of Hamel functions

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Basics

- A subset H of $\mathbb{R}$ ($\mathbb{R}^2$) is called a **Hamel basis** if it is a basis of $\mathbb{R}$ ($\mathbb{R}^2$) over $\mathbb{Q}$.

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$\mathrm{HF} + \mathrm{HF} = \mathbb{R}^{\mathbb{R}}$.

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The goal

Definition

We say that a group $(G, \star)$ is **free** if there exist a set $S \subset G$ of free generators: every element of G can be expressed in exactly one reduced way using generators ($a^2 \star a^3$, $a \star a^{-1}$ are not in reduced form).

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Elements of a free group are called **words**.

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${}_\xi f_0$	${}_\xi f_1$	....	${}_\xi f_\gamma$	....
... ...	... ...	... ...	... ...	... ...
$f_0 := \bigcup_{\alpha<\mathfrak{c}} {}_\alpha f_0$	$f_1 := \bigcup_{\alpha<\mathfrak{c}} {}_\alpha f_1$	....	$f_\gamma := \bigcup_{\alpha<\mathfrak{c}} {}_\alpha f_\gamma$	....

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Observation

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Indeed, let $\langle x, y \rangle \in \mathbb{R}^2$. Then

$$\langle x, y \rangle = \langle 0, y - f(x) \rangle + \langle x, f(x) \rangle .$$

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Now consider a set W of all the reduced words that can be composed from the generators, i. e. functions of the form

$$h = f_{\gamma_1}^{k_1} \circ \ldots \circ f_{\gamma_m}^{k_m}.$$

where $m \geqslant 1$, $k_i \in \mathbb{Z} \setminus \{0\}$, $\gamma_i < \mathfrak{c}$ and $\gamma_i \neq \gamma_{i+1}$.

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$$W = \{h_\alpha : \alpha < \mathfrak{c}\}.$$

If $h_\alpha = f_{\gamma_1}^{k_1} \circ \ldots \circ f_{\gamma_m}^{k_m}$ then by ${}_\xi h_\alpha$ we will denote

$${}_\xi f_{\gamma_1}^{k_1} \circ \ldots \circ_\xi f_{\gamma_m}^{k_m},$$

i. e. the word h_α at the ξ-stage of the construction.

Conditions

For every $\beta < \mathfrak{c}$ (number of the generator/word) and for every $\kappa < \mathfrak{c}$ (number of the stage of construction):

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For every $\beta < \mathfrak{c}$ (number of the generator/word) and for every $\kappa < \mathfrak{c}$ (number of the stage of construction):

(I) ${}_\kappa f_\beta$ is a partial function (has at most one value in every $x \in \mathbb{R}$);

(II) ${}_\kappa f_\beta$ is one-to-one;

(III) ${}_\xi f_\beta \subset_\kappa f_\beta$ for $\xi < \kappa$;

(IV) $|\bigcup_{\gamma<\beta} {}_\kappa f_\gamma| \leqslant |\kappa| + \omega$;

(V) ${}_\kappa h_\beta \in \mathrm{PLIF}$;

(VI) $\langle 0, x_\kappa \rangle \in \mathrm{LIN}_{\mathbb{Q}}({}_{\kappa+1} h_{\alpha_\kappa})$;

(VII) $x_\kappa \in \mathrm{dom}({}_{\kappa+1} f_{\alpha_\kappa})$;

(VIII) $x_\kappa \in \mathrm{rng}({}_{\kappa+1} f_{\alpha_\kappa})$.

At the end for every $\beta < \mathfrak{c}$ let

$$f_\beta := \bigcup_{\kappa<\mathfrak{c}} {}_\kappa f_\beta.$$

Why do these conditions suffice?

These conditions assure that for every $\beta < \mathfrak{c}$, $f_\beta \in \mathbb{R}^{\mathbb{R}}$.

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These conditions assure that every word is a Hamel basis.

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Indeed, if it was not, some function would have two representations that do not reduce. Composing the function with its inverse would lead to a nontrivial representation of the identity function, a contradiction.

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We will see that this condition will enable us to make the inductive step.

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How do we care about them?

The highlighted conditions are true from the very beginning of our construction. We just need to make sure we don't break any of these.

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On the other hand, conditions (VI)-(VII) are the conditions that we need to make work.

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Construction

Assume that for each β (number of generator) ${}_{\xi}f_{\beta}$ are constructed for $\xi < \eta$. If η is a limit cardinal then for each β we let

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STEP I

In this step we make sure that (VI) holds.
If $\langle 0, x_\kappa \rangle \in \mathrm{LIN}_{\mathbb{Q}}({}_\kappa h_{\alpha_\kappa})$, we don't change anything. Let's look at the other case.

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First we choose x linearly independent of F. Then we choose y independent of $F \cup \{x\}$.

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First we choose x linearly independent of F. Then we choose y independent of $F \cup \{x\}$. Then we have to choose

$$ 2 \cdot \sum |k_i| $$

pairs of points and add them to appropiate f_{γ_i}'s in the way that $\langle x, y \rangle, \langle -x, x_\kappa - y \rangle$ are in the extended ${}_{\kappa}h_{\alpha_\kappa}$.

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pairs of points and add them to appropiate f_{γ_i}'s in the way that $\langle x, y \rangle, \langle -x, x_\kappa - y \rangle$ are in the extended ${}_{\kappa}h_{\alpha_\kappa}$. It is easy to check that conditions (I)-(IV) still hold. (V) remains true because we were chosing point that were linearly independent.

STEP II and STEP III

In these steps we have to make conditions (VII) and (VIII) hold. The argument showing that it can be done without breaking conditions (I)-(V) is the same - the set of "forbidden" point is not equal to $\mathbb{R}$.

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In these steps we have to make conditions (VII) and (VIII) hold. The argument showing that it can be done without breaking conditions (I)-(V) is the same - the set of "forbidden" point is not equal to $\mathbb{R}$.
At the end we let ${}_{\kappa+1}f_\beta$ be the extended version of ${}_\kappa f_\beta$ or ${}_{\kappa+1}f_\beta =_\kappa f_\beta$ if it was not changed in steps I-III.

Problems

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Question to the audience

Do you know other examples of large free groups within some structures?

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Thank you for your attention!

References

- G. Matusik, T. Natkaniec, Algebraic properties of Hamel functions, Acta Math. Hungar., 126 (3), 2010, 209-229.
- K. Płotka, On functions whose graph is a Hamel basis, Proc. Amer. Math. Soc., 131, 2003, 1031-1041.