Special Families of Magic Sets

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Winter School in Abstract Analysis, Hejnice, Febuery 2022

Cinderella

Definition (Berarducci, Dikranjan, 1993)

Given a family of functions $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ we say that set $M \subset \mathbb{R}$ is *magic* for $\mathcal{F}$ if for any $f, g \in \mathcal{F}$ we have

$$f[M] \subset g[M] \implies f = g.$$

Equivalently

$$f \neq g \implies f[M] \not\subset g[M]$$

Result by Berarducci and Dikranjan

Theorem

Assume $\text{add}(\mathcal{M}) = \mathfrak{c}$. There exists $2^{\mathfrak{c}}$ many different magic sets for the family $\mathcal{F}$-continuous, nowhere constant.

Sketch of the proof

Enumerate all pairs $(f, g), f \neq g, f, g \in \mathcal{F}$.

By transfinite induction we construct a magic set in the following way

- we find x_0 for pair (f_0, g_0) such that $f_0(x_0) \neq g_0(x_0)$
- we find x_1 for pair (f_1, g_1) such that $f_1(x_1) \neq g_1(x_1)$

  And... we have a problem.

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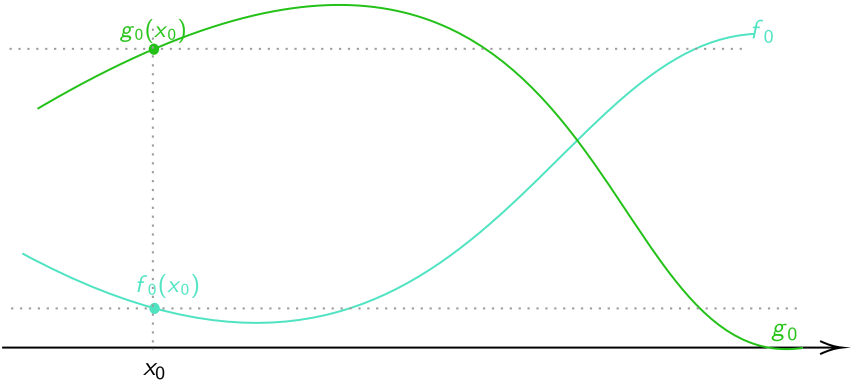
problem



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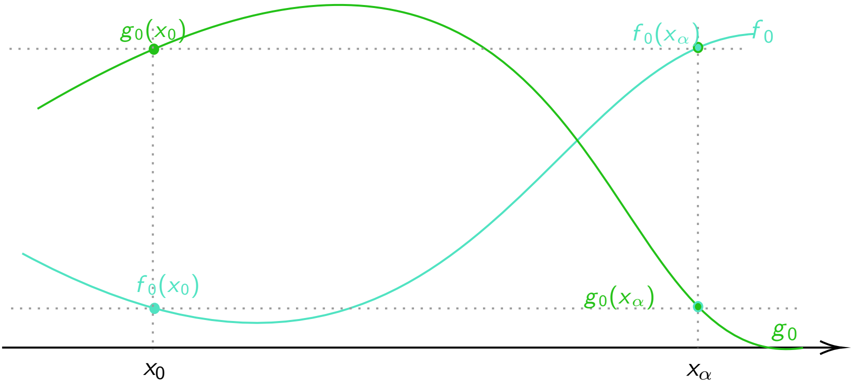

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- all x's are different,
- $f_\alpha(x_\alpha) \neq g_\alpha(x_\alpha)$
- $x_\alpha^0, x_\alpha^1 \notin A_\alpha \cup B_\alpha$, where

$A_\alpha = \bigcup_{\beta<\alpha} f_\beta^{-1}[\{g_\beta(x_\beta)\}] \cup g_\beta^{-1}[\{f_\beta(x_\beta)\}]$
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$\forall\ \varphi : \mathfrak{c} \to 2$ let $M_\varphi := \{x_\eta^{\varphi(\eta)} : \eta < \mathfrak{c}\}$. All M_φ's are magic for $\mathcal{F}$.

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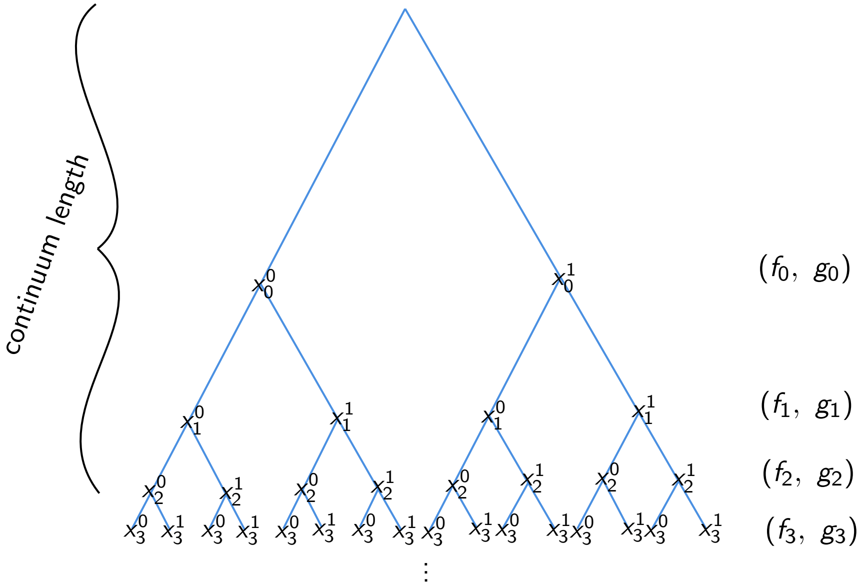
Loooong tree



Independent family

Independent family

A family $\mathcal{A} \subseteq \mathcal{P}(X)$ of subsets of X is called independent if whenever we have distinct $A_1, \ldots, A_n, B_1, \ldots, B_m \in \mathcal{A}$ then

$$A_1 \cap \ldots \cap A_n \cap (X \setminus B_1) \cap \ldots \cap (X \setminus B_m) \neq \emptyset.$$

Fichtenholz-Kantorowicz Theorem

For every set of carnality $\kappa \geqslant \aleph_0$ There exists a independent family of its subset of carnality 2^κ.

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Thank you for your attention!