Continuity of coordinate $\mathcal{I}$-projections without large cardinals

Jarosław Swaczyna

Łódź University of Technology & Czech Academy of Sciences

Winter School on Abstract Analysis 2022, section Set Theory & Topology, Hejnice

joint work with Noé de Rancourt and Tomasz Kania

The author acknowledge with thanks support received from GACR project 19-07129Y; RVO 67985840.

Ideal convergence

Definition

We say that a sequence (x_n) is $\mathcal{I}$-convergent to x if for every $\varepsilon > 0$ we have $\{n \in \mathbb{N} : d(x, x_n) > \varepsilon\} \in \mathcal{I}$.

Observation

For the ideal *Fin* $\mathcal{I}$-convergence is just the classical one.

Ideal convergence

Definition

We say that a sequence (x_n) is $\mathcal{I}$-convergent to x if for every $\varepsilon > 0$ we have $\{n \in \mathbb{N} : d(x, x_n) > \varepsilon\} \in \mathcal{I}$.

Observation

For the ideal *Fin* $\mathcal{I}$-convergence is just the classical one.

$\mathcal{I}$-Schauder basis

Definition

Given an ideal $\mathcal{I}$ on ω, we say that a sequence (e_n) is $\mathcal{I}$-basis if for every $x \in X$ there exists a unique sequence $(\alpha_n) \in \mathbb{K}^\omega$ such that $x = \sum_{n,\mathcal{I}} \alpha_n e_n$. We denote the coordinate functionals by $e_n^\star$ and we set $P_n := \sum_{i=1}^n e_i^\star e_i$.

Question (Kadets)

Are $e_n^\star$ continuous for the $\mathcal{I}$ basis? At least for nice filters, e.g. $\mathcal{I}_{st}$?

$\mathcal{I}$-Schauder basis

Definition

Given an ideal $\mathcal{I}$ on ω, we say that a sequence (e_n) is $\mathcal{I}$-basis if for every $x \in X$ there exists a unique sequence $(\alpha_n) \in \mathbb{K}^\omega$ such that $x = \sum_{n,\mathcal{I}} \alpha_n e_n$. We denote the coordinate functionals by $e_n^\star$ and we set $P_n := \sum_{i=1}^n e_i^\star e_i$.

Question (Kadets)

Are $e_n^\star$ continuous for the $\mathcal{I}$ basis? At least for nice filters, e.g. $\mathcal{I}_{st}$?

$\mathcal{I}$-Schauder basis

Definition

Given an ideal $\mathcal{I}$ on ω, we say that a sequence (e_n) is $\mathcal{I}$-basis if for every $x \in X$ there exists a unique sequence $(\alpha_n) \in \mathbb{K}^\omega$ such that $x = \sum_{n,\mathcal{I}} \alpha_n e_n$. We denote the coordinate functionals by $e_n^\star$ and we set $P_n := \sum_{i=1}^n e_i^\star e_i$.

Question (Kadets)

Are $e_n^\star$ continuous for the $\mathcal{I}$ basis? At least for nice filters, e.g. $\mathcal{I}_{st}$?

We consider the space $S := \{(\alpha_n) \in \mathbb{K}^\omega : \sum_{n=0}^{\infty} \alpha_n e_n$ is convergent$\}$ equipped with the norm $|||(|||\alpha_n)|||=|||\sup_{n\omega} \| \sum_{i=0}^{n} \alpha_i e_i \|$, and map $T : S \to X$ given by $T((\alpha_n)) = \sum_{n=0}^{\infty} \alpha_n e_n$. Clearly T is a bijection. It is also continuous. Now it remains to prove that S is a Banach space, and use the open mapping principle. Byproduct: the norms of projections have common bound.

Proof for $\mathcal{I} = Fin$

We consider the space $S := \{(\alpha_n) \in \mathbb{K}^\omega : \sum_{n=0}^\infty \alpha_n e_n$ is convergent$\}$ equipped with the norm $|||(|||\alpha_n)|||=|||\sup_{n\,\omega} \| \sum_{i=0}^n \alpha_i e_i \|$, and map $T\colon S \to X$ given by $T((\alpha_n)) = \sum_{n=0}^\infty \alpha_n e_n$. Clearly T is a bijection. It is also continuous. Now it remains to prove that S is a Banach space, and use the open mapping principle. Byproduct: the norms of projections have common bound.

Proof for $\mathcal{I} = Fin$

We consider the space $S := \{(\alpha_n) \in \mathbb{K}^\omega : \sum_{n=0}^{\infty} \alpha_n e_n$ is convergent$\}$ equipped with the norm $|||(|||\alpha_n)|||=||| \sup_{n\omega} \| \sum_{i=0}^{n} \alpha_i e_i \|$, and map $T \colon S \to X$ given by $T((\alpha_n)) = \sum_{n=0}^{\infty} \alpha_n e_n$. Clearly T is a bijection. It is also continuous. Now it remains to prove that S is a Banach space, and use the open mapping principle. Byproduct: the norms of projections have common bound.

Proof for $\mathcal{I} = Fin$

We consider the space
$S := \{(\alpha_n) \in \mathbb{K}^\omega : \sum_{n=0}^{\infty} \alpha_n e_n$ is convergent$\}$ equipped with the norm $|||(|||\alpha_n)|||=|||\sup_{n\,\omega} \| \sum_{i=0}^{n} \alpha_i e_i \|$, and map $T\colon S \to X$ given by $T((\alpha_n)) = \sum_{n=0}^{\infty} \alpha_n e_n$. Clearly T is a bijection. It is also continuous. Now it remains to prove that S is a Banach space, and use the open mapping principle. Byproduct: the norms of projections have common bound.

Proof for $\mathcal{I} = Fin$

We consider the space $S := \{(\alpha_n) \in \mathbb{K}^\omega : \sum_{n=0}^{\infty} \alpha_n e_n$ is convergent$\}$ equipped with the norm $|||(|||\alpha_n)|||=||| \sup_{n\,\omega} \| \sum_{i=0}^{n} \alpha_i e_i \|$, and map $T\colon S \to X$ given by $T((\alpha_n)) = \sum_{n=0}^{\infty} \alpha_n e_n$. Clearly T is a bijection. It is also continuous. Now it remains to prove that S is a Banach space, and use the open mapping principle. Byproduct: the norms of projections have common bound.

Proof for $\mathcal{I} = Fin$

We consider the space $S := \{(\alpha_n) \in \mathbb{K}^\omega : \sum_{n=0}^\infty \alpha_n e_n$ is convergent$\}$ equipped with the norm $|||(|||\alpha_n)|||=|||\sup_{n\omega}\|\sum_{i=0}^n \alpha_i e_i\|$, and map $T\colon S \to X$ given by $T((\alpha_n)) = \sum_{n=0}^\infty \alpha_n e_n$. Clearly T is a bijection. It is also continuous. Now it remains to prove that S is a Banach space, and use the open mapping principle. Byproduct: the norms of projections have common bound.

Problems with classical proof

We consider the space ℓ_2 and we let $x_n = \sum_{i=1}^{n} e_i$, where (e_n) stands for standard basis. Sequence (x_n) is a $\mathcal{I}_{st}$ basis, but projections P_n related to it are not uniformly bounded.

The standard proof will not work.

Partial answer (Kochanek 2012)

If $\mathcal{I}$ is an ideal generated by less than $\mathfrak{p}$ sets, then the coordinate projections associated with $\mathcal{I}$-basis are continuous.

Problems with classical proof

We consider the space ℓ_2 and we let $x_n = \sum_{i=1}^{n} e_i$, where (e_n) stands for standard basis. Sequence (x_n) is a $\mathcal{I}_{st}$ basis, but projections P_n related to it are not uniformly bounded.
The standard proof will not work.

Partial answer (Kochanek 2012)

If $\mathcal{I}$ is an ideal generated by less than $\mathfrak{p}$ sets, then the coordinate projections associated with $\mathcal{I}$-basis are continuous.

Problems with classical proof

We consider the space ℓ_2 and we let $x_n = \sum_{i=1}^{n} e_i$, where (e_n) stands for standard basis. Sequence (x_n) is a $\mathcal{I}_{st}$ basis, but projections P_n related to it are not uniformly bounded.
The standard proof will not work.

Partial answer (Kochanek 2012)

If $\mathcal{I}$ is an ideal generated by less than $\mathfrak{p}$ sets, then the coordinate projections associated with $\mathcal{I}$-basis are continuous.

Outdated (from two weeks) proof - large cardinals

Theorem

We assume enough large cardinals to get that

- every subset of $\mathbb{R}$ that is in $L(\mathbb{R})$ has the Baire property, (Shelah-Woodin)
- in $L(\mathbb{R})$ every linear map between Fréchet spaces (in particular, Banach spaces) is continuous (Garnir, Wright)
- every projective formula is absolute between V and $L(\mathbb{R})$

Outdated (from two weeks) proof - the space SB

Let $\mathcal{F}(C(\Delta))$ denote the hyperspace comprising all non-empty closed subsets of $C(\Delta)$.

Following Godefroy and Saint-Raymond, we shall call a Polish topology τ on $\mathcal{F}(C(\Delta))$ *admissible*, whenever

- $E^+(U) \in \tau$ for every open set $U \subseteq C(\Delta)$,
- there is a subbase $\mathcal{B}$ of τ such that every set $U \in \mathcal{B}$ may be written as a union of countably many sets of the form $E^+(U) \setminus E^+(V)$, where U and V are open in $C(\Delta)$.

It turns out that the set SB comprising all closed linear subspaces of $C(\Delta)$ is Π^0_2 in $\mathcal{F}(C(\Delta))$ and, as such, the relative topology on SB is Polish. Recently some other approaches to the universal space for separable Banach spaces was made (see eg paper by Cúth, Doležal, Doucha and Kurka).

Outdated (from two weeks) proof - the space SB

Let $\mathcal{F}(C(\Delta))$ denote the hyperspace comprising all non-empty closed subsets of $C(\Delta)$.
Following Godefroy and Saint-Raymond, we shall call a Polish topology τ on $\mathcal{F}(C(\Delta))$ *admissible*, whenever

- $E^+(U) \in \tau$ for every open set $U \subseteq C(\Delta)$,
- there is a subbase $\mathcal{B}$ of τ such that every set $U \in \mathcal{B}$ may be written as a union of countably many sets of the form $E^+(U) \setminus E^+(V)$, where U and V are open in $C(\Delta)$.

It turns out that the set SB comprising all closed linear subspaces of $C(\Delta)$ is Π^0_2 in $\mathcal{F}(C(\Delta))$ and, as such, the relative topology on SB is Polish. Recently some other approaches to the universal space for separable Banach spaces was made (see eg paper by Cúth, Doležal, Doucha and Kurka).

Outdated (from two weeks) proof - the space *SB*

Let $\mathcal{F}(C(\Delta))$ denote the hyperspace comprising all non-empty closed subsets of $C(\Delta)$.
Following Godefroy and Saint-Raymond, we shall call a Polish topology τ on $\mathcal{F}(C(\Delta))$ *admissible*, whenever

- $E^+(U) \in \tau$ for every open set $U \subseteq C(\Delta)$,
- there is a subbase $\mathcal{B}$ of τ such that every set $U \in \mathcal{B}$ may be written as a union of countably many sets of the form $E^+(U) \setminus E^+(V)$, where U and V are open in $C(\Delta)$.

It turns out that the set SB comprising all closed linear subspaces of $C(\Delta)$ is Π^0_2 in $\mathcal{F}(C(\Delta))$ and, as such, the relative topology on SB is Polish. Recently some other approaches to the universal space for separable Banach spaces was made (see eg paper by Cúth, Doležal, Doucha and Kurka).

Outdated (from two weeks) proof - the space *SB*

Let $\mathcal{F}(C(\Delta))$ denote the hyperspace comprising all non-empty closed subsets of $C(\Delta)$.
Following Godefroy and Saint-Raymond, we shall call a Polish topology τ on $\mathcal{F}(C(\Delta))$ *admissible*, whenever

- $E^+(U) \in \tau$ for every open set $U \subseteq C(\Delta)$,
- there is a subbase $\mathcal{B}$ of τ such that every set $U \in \mathcal{B}$ may be written as a union of countably many sets of the form $E^+(U) \setminus E^+(V)$, where U and V are open in $C(\Delta)$.

It turns out that the set SB comprising all closed linear subspaces of $C(\Delta)$ is Π^0_2 in $\mathcal{F}(C(\Delta))$ and, as such, the relative topology on SB is Polish. Recently some other approaches to the universal space for separable Banach spaces was made (see eg paper by Cúth, Doležal, Doucha and Kurka).

Outdated Main Theorem

Theorem (Kania, S.)

Under LC the coordinate functionals of $\mathcal{I}$ basis are continuous for any projective filter $\mathcal{I}$ on $\mathbb{N}$.

Main proof

$$\forall_{X\in \mathrm{SB}}\forall_{(x_k)_{k=1}^{\infty}\in X^{\mathbb{N}}}\Big[\neg\Big(\forall_{y\in X}\ \exists!_{(a_k)_{k=1}^{\infty}\in\mathbb{K}^{\mathbb{N}}}\ \sum_{k,\mathcal{F}} a_k x_k = y\Big)\vee$$
$$\vee\Big(\exists_{(M_k)_{k=1}^{\infty}\in\mathbb{N}^{\mathbb{N}}}\ \forall_{y\in X}\ \exists_{(a_k)_{k=1}^{\infty}\in\mathbb{K}^{\mathbb{N}}}\ \sum_{k,\mathcal{F}} a_k x_k = y \wedge |a_k| \leq \|y\|\cdot M_k\Big)\Big].$$

Outdated Main Theorem

Theorem (Kania, S.)

Under LC the coordinate functionals of $\mathcal{I}$ basis are continuous for any projective filter $\mathcal{I}$ on $\mathbb{N}$.

Lemma

Let X be a separable Banach space and let $\mathcal{I}$ be a projective filter on $\mathbb{N}$ of class $\prod_n^1$. Suppose that $(z_k)_{k=1}^\infty$ is a sequence in X. Then, the following formula is $\prod_n^1$:

$$\varphi((a_k)_{k=1}^\infty, z) \equiv \sum_{j,\mathcal{I}} a_k z_k = z.$$

Main proof

$$\forall_{X \in \mathrm{SB}} \forall_{(x_k)_{k=1}^\infty \in X^{\mathbb{N}}} \Big[\neg \big(\forall_{y \in X} \exists!_{(a_k)_{k=1}^\infty \in \mathbb{K}^{\mathbb{N}}} \sum_{k,\mathcal{F}} a_k x_k = y \big) \vee$$

Outdated Main Theorem

Theorem (Kania, S.)

Under LC the coordinate functionals of $\mathcal{I}$ basis are continuous for any projective filter $\mathcal{I}$ on $\mathbb{N}$.

Main proof

$$\forall_{X \in \mathrm{SB}} \forall_{(x_k)_{k=1}^\infty \in X^{\mathbb{N}}} \Big[\neg \Big(\forall_{y \in X} \, \exists!_{(a_k)_{k=1}^\infty \in \mathbb{K}^{\mathbb{N}}} \sum_{k,\mathcal{F}} a_k x_k = y \Big) \vee$$
$$\vee \Big(\exists_{(M_k)_{k=1}^\infty \in \mathbb{N}^{\mathbb{N}}} \, \forall_{y \in X} \, \exists_{(a_k)_{k=1}^\infty \in \mathbb{K}^{\mathbb{N}}} \sum_{k,\mathcal{F}} a_k x_k = y \wedge |a_k| \leq \|y\| \cdot M_k \Big) \Big].$$

Modern results

Main Theorem

Let $\mathcal{I}$ be analytic filter on $\mathbb{N}$. Then for every $\mathcal{I}$-basis of a Banach space the corresponding coordinate functionals are continuous.

Proof

$$e_n^\star(x) \in U \Leftrightarrow \exists_{(\alpha_i) \in \mathbb{K}^\mathbb{N}} \sum_{i,\mathcal{I}} \alpha_i e_i = x \wedge \alpha_n \in U$$

$$\Leftrightarrow \exists_{(\alpha_i) \in \mathbb{K}^\mathbb{N}} \forall_{l \in \mathbb{N}} \exists_{A \in \mathcal{I}} \forall_{m \notin A} \left\| \sum_{i=1}^{m} \alpha_i e_i - x \right\| \leq \frac{1}{l} \wedge \alpha_n \in U$$

$$e_n^\star(x) \in U \Leftrightarrow \forall_{b \in \mathbb{K}} (b \in U) \vee e_n^\star(x) \neq b$$

Modern results

Main Theorem

Let $\mathcal{I}$ be analytic filter on $\mathbb{N}$. Then for every $\mathcal{I}$-basis of a Banach space the corresponding coordinate functionals are continuous.

Proof

$$e_n^\star(x) \in U \Leftrightarrow \exists_{(\alpha_i)\in\mathbb{K}^\mathbb{N}} \sum_{i,\mathcal{I}} \alpha_i e_i = x \wedge \alpha_n \in U$$

$$\Leftrightarrow \exists_{(\alpha_i)\in\mathbb{K}^\mathbb{N}} \forall_{l\in\mathbb{N}} \exists_{A\in\mathcal{I}} \forall_{m\notin A} \left\| \sum_{i=1}^{m} \alpha_i e_i - x \right\| \leq \frac{1}{l} \wedge \alpha_n \in U$$

$$e_n^\star(x) \in U \Leftrightarrow \forall_{b\in\mathbb{K}} (b \in U) \vee e_n^\star(x) \neq b$$

Modern results

Main Theorem

Let $\mathcal{I}$ be analytic filter on $\mathbb{N}$. Then for every $\mathcal{I}$-basis of a Banach space the corresponding coordinate functionals are continuous.

Proof

$$e_n^\star(x) \in U \Leftrightarrow \exists_{(\alpha_i)\in\mathbb{K}^\mathbb{N}} \sum_{i,\mathcal{I}} \alpha_i e_i = x \wedge \alpha_n \in U$$

$$\Leftrightarrow \exists_{(\alpha_i)\in\mathbb{K}^\mathbb{N}} \forall_{l\in\mathbb{N}} \exists_{A\in\mathcal{I}} \forall_{m\notin A} \left\| \sum_{i=1}^{m} \alpha_i e_i - x \right\| \leq \frac{1}{l} \wedge \alpha_n \in U$$

$$e_n^\star(x) \in U \Leftrightarrow \forall_{b\in\mathbb{K}} (b \in U) \vee e_n^\star(x) \neq b$$

Modern results

Main Theorem

Let $\mathcal{I}$ be analytic filter on $\mathbb{N}$. Then for every $\mathcal{I}$-basis of a Banach space the corresponding coordinate functionals are continuous.

Theorem

Assume that all Δ^1_n-sets are Baire-measureable. Let $\mathcal{F}$ be Σ^1_n-ideal on ω. Then for every $\mathcal{I}$-basis the corresponding coordinate functionals are continuous.

Theorem

Let $\mathcal{I}$ by an ideal on ω (not necessarily projective). Let (e_n) be an $\mathcal{I}$-basis with continuous coordinate functionals. Then there exists an analytic ideal $\mathcal{I}' \subset \mathcal{I}$ on ω such that (x_n) is also an $\mathcal{I}'$-basis.

Main Theorem

Let $\mathcal{I}$ be analytic filter on $\mathbb{N}$. Then for every $\mathcal{I}$-basis of a Banach space the corresponding coordinate functionals are continuous.

Theorem

Assume that all Δ^1_n-sets are Baire-measureable. Let $\mathcal{F}$ be Σ^1_n-ideal on ω. Then for every $\mathcal{I}$-basis the corresponding coordinate functionals are continuous.

Theorem

Let $\mathcal{I}$ by an ideal on ω (not necessarily projective). Let (e_n) be an $\mathcal{I}$-basis with continuous coordinate functionals. Then there exists an analytic ideal $\mathcal{I}' \subset \mathcal{I}$ on ω such that (x_n) is also an $\mathcal{I}'$-basis.

Question

In the paper

J. Humkins, H. Woodin, Small forcing creates neither strong nor Woodin cardinals, *Proc. Amer. Math. Soc.* **128** (2000), 3025–3029

We found the following result:

"After small forcing, a cardinal θ is Woodin if and only if it was Woodin in the ground model".

What does it mean that forcing is small? It looks like "of cardinality less than θ" makes sense, but if somebody knows for sure please let me know.

Question

In the paper
J. Humkins, H. Woodin, Small forcing creates neither strong nor Woodin cardinals, *Proc. Amer. Math. Soc.* **128** (2000), 3025–3029
We found the following result:
"After small forcing, a cardinal θ is Woodin if and only if it was Woodin in the ground model".
What does it mean that forcing is small? It looks like "of cardinality less than θ" makes sense, but if somebody knows for sure please let me know.

Bibliography

Noé de Rancourt, Tomasz Kania, Jarosław Swaczyna, Continuity of coordinate functionals of filter bases in Banach spaces, preprint

Tomasz Kania, Jarosław Swaczyna, Large cardinals and continuity of coordinate functionals of filter bases in Banach spaces, Bull. London Math. Soc., 53 (1) (2021), 231-239

T. Kochanek, $\mathcal{F}$-bases with brackets and with individual brackets in Banach spaces, *Studia Math.* **211** (2012), 259–268.

J. Connor, M. Ganichev, and V. Kadets, A characterization of Banach spaces with separable duals via weak statistical convergence, *J. Math. Anal. Appl.* **244** (2000), 251–261.

Thank you for your attention!
Gratiam vobis ago pro animis attentis!
Σας ευχαριστώ για την προσοχή σας!
Dziękuję za uwagę! Děkuji za pozornost!
Danke für Ihre Aufmerksamkeit!
Grazie per l'attenzione! Merci de votre attention !
Ďakujem za vašu pozornosť!
Gracias por su atención! תודה לך על תשומת הלב Bedankt voor uw aandacht! Спасибо за внимание!