The tree ideal of full-splitting Miller trees

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Tree ideals

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If $\mathbb{P}$ is some collection of trees, like Sacks, Miller, Laver, etc. the tree ideal p_0 consists of $X \subseteq 2^\omega$ or ω^ω, such that

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- s_0 - Marczewski ideal
- m_0, l_0 - Miller and Laver ideal
- v_0 - Silver, Mycielski ideal

Full-Miller trees

Definition

A tree $p \subseteq \omega^{<\omega}$ is full-Miller if every $\sigma \in p$ has an extension $\tau \in p$, $\sigma \subseteq \tau$ which splits fully i.e. $\forall n \in \omega \; \tau^\frown n \in p$

with fm_0 as corresponding tree ideal

Borel part of fm_0

For classical tree types the following perfect set style theorem holds

Theorem

For every $A \in \Sigma^1_1$ we have:

$$\exists p \in \mathbb{P}\ [p] \subseteq A \text{ or } A \in \mathcal{I}_{\mathbb{P}}$$

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Theorem (Newelski, Rosłanowski)

$\mathcal{I}_{\mathbb{FM}}$ is σ-ideal generated by sets of form:

$$D_\phi = \{x \in \omega^\omega : \forall^\infty n\ x(n) \neq \phi(x|_n)\} \text{ with } \phi : \omega^{<\omega} \to \omega$$

Additivity and Covering of fm_0

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The following inequalities hold

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Fix enumeration $2^{<\omega} = \{\sigma_n : n < \omega\}$ and define function $\phi : \omega^\omega \to 2^\omega$ for $x =< x_n : n < \omega >$ to be:

$$\phi(x) = \sigma_{x_0} {}^\frown \sigma_{x_1} {}^\frown \sigma_{x_2} {}^\frown \ldots$$

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Next, we define $\Phi : \mathcal{M} \to fm_0$, for $M \in \mathcal{M}$ to be

$$\Phi(M) = \{x \in \omega^\omega : \phi(x) \in M\}$$

Cofinality of fm_0

Theorem (Brendle, Khomskii, Wohofsky)

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$$\forall p \in \mathbb{P}\ \forall \{q_\alpha : \alpha < \kappa\} \subseteq \mathbb{P} \text{ with } \kappa < \mathfrak{c}$$
$$(\text{if } \forall \alpha < \kappa\ p \perp p_\alpha \text{ then } \exists q \leq p\ \forall \alpha < \kappa\ [q] \cap [p_\alpha] = \varnothing)$$

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Define partial ordering

$$P = \{F \subseteq p : F \text{ is finite tree}\}$$

P is countable so recall that $cov(\mathcal{M})$ is equal to the smallest number of dense subsets of any countable poset for which there is no filter intersecting them all.

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For each $\alpha < \kappa$ we have $[p] \cap [p_\alpha] \subseteq D_{\phi_\alpha}$ and we define dense set

$$A_\alpha = \{F \in P : \forall \sigma \in ter(F))\ \sigma(|\sigma| - 1) = \phi_\alpha(\sigma|_n)\}$$

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As $\kappa < cov(\mathcal{M})$ there exists filter $H \subseteq P$ which hits all of A_α's.
It follows that if we define $q = \bigcup H$ we are guarantied that $q \leq p$ is full-splitting Miller tree and $[q] \cap D_{\phi_\alpha} = \emptyset$ for each $\alpha < \kappa$

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Given any $\{p_n : n \in \omega\} \subseteq \mathbb{FM}$ and any $c \Vdash (q \in \mathbb{FM},\ x \in \omega^\omega)$ such that $\forall n \in \omega\ c \Vdash (q$ and p_n are incompatible) we can find $p \in \mathbb{FM}$ incompatible with each p_n and such that $c \Vdash (q$ is compatible with $p)$ and $c \Vdash x \notin [p]$.

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Define full-Miller amoeba $\mathbb{A}(\mathbb{FM})$ as set of pairs (F, p) where $p \in \mathbb{FM}$ and $F \subseteq p$ is finite tree. Forcing $\mathbb{A}(\mathbb{FM})$ adds full-Miller tree each branch of is full-Miller generic real.

Thank you