Fullness and mixing property for boolean valued models in terms of sheaves and bundles
joint work with Matteo Viale

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Boolean algebras

Given a topological space X, let $\mathrm{CLOP}(X)$ be the boolean algebra of the clopen subsets of X.

The Stone space $\mathrm{St}(\mathrm{B})$ of a boolean algebra B is

$$\mathrm{St}(\mathrm{B}) := \{G : G \textit{ is an ultrafilter of } \mathrm{B}\}.$$

The base for the topology is:

$$\{N_b := \{G \in \mathrm{St}(\mathrm{B}) : b \in G\} : b \in \mathrm{B}\}.$$

B is isomorphic to $\mathrm{CLOP}(\mathrm{St}(\mathrm{B}))$ via the Stone duality map

$$b \mapsto N_b = \{G \in \mathrm{St}(\mathrm{B}) : b \in G\}$$

Boolean completions

If X is a topological space and $A \subset X$, $\mathrm{Reg}\,(A)$ is the interior of the closure of A in X. A is *regular open* if $A = \mathrm{Reg}\,(A)$.
$\mathrm{RO}(X)$ is the family of regular open subsets of X ($CLOP(X) \subseteq \mathrm{RO}(X)$).

$\mathrm{RO}(X)$ *is a complete boolean algebra, with the operations given by*

$$\neg U = X \setminus \overline{U}, \quad \bigvee_{i \in I} U_i := Reg\left(\bigcup_{i \in I} U_i\right), \quad \bigwedge_{i \in I} U_i := Reg\left(\bigcap_{i \in I} U_i\right).$$

A boolean algebra B is complete if and only if $\mathsf{CLOP}(\mathsf{St}(\mathsf{B})) = \mathsf{RO}(\mathsf{St}(\mathsf{B}))$.

Every boolean algebra B can be densely embedded in the complete boolean algebra $\mathsf{RO}(\mathsf{St}(\mathsf{B}))$ via the Stone duality map.

Boolean valued models

Definition

Let B be a *boolean algebra* and $\mathcal{L}$ be a first order *relational* language.
A B-*valued model* for $\mathcal{L}$ is a tuple
$\mathcal{M} = \langle M, =^{\mathcal{M}}, R_i^{\mathcal{M}} : i \in I, c_j^{\mathcal{M}} : j \in J \rangle$
with

$$=^{\mathcal{M}}: M^2 \to \mathsf{B}$$
$$(\tau, \sigma) \mapsto [\![\tau = \sigma]\!]_{\mathsf{B}}^{\mathcal{M}} = [\![\tau = \sigma]\!],$$

$$R^{\mathcal{M}} : M^n \to \mathsf{B}$$
$$(\tau_1, \ldots, \tau_n) \mapsto [\![R(\tau_1, \ldots, \tau_n)]\!]_{\mathsf{B}}^{\mathcal{M}} = [\![R(\tau_1, \ldots, \tau_n)]\!]$$

for $R \in \mathcal{L}$ an n-ary relation symbol.

The constraints on $R^{\mathcal{M}}$ and $=^{\mathcal{M}}$ are the following:

- for $\tau, \sigma, \chi \in M$,
  1. $[\![\tau = \tau]\!] = 1_{\mathsf{B}}$;
  2. $[\![\tau = \sigma]\!] = [\![\sigma = \tau]\!]$;
  3. $[\![\tau = \sigma]\!] \wedge [\![\sigma = \chi]\!] \leq [\![\tau = \chi]\!]$;
- for $R \in \mathcal{L}$ with arity n, and $(\tau_1, \ldots, \tau_n), (\sigma_1, \ldots, \sigma_n) \in M^n$,

$$[\![R(\tau_1, \ldots, \tau_n)]\!] \wedge \bigwedge_{h \in \{1,\ldots,n\}} [\![\tau_h = \sigma_h]\!] \leq [\![R(\sigma_1, \ldots, \sigma_n)]\!] .$$

Definition

Let $\mathcal{M}$ be a B-valued model in the relational language $\mathcal{L}$. The *boolean value*

$$[\![\phi]\!]^{\mathcal{M}}_{\mathsf{B}} = [\![\phi]\!]$$

of ϕ is defined by recursion as follows:

- $[\![\neg\psi]\!] = \neg [\![\psi]\!]$;
- $[\![\psi \wedge \theta]\!] = [\![\psi]\!] \wedge [\![\theta]\!]$;
- $[\![\exists y \psi(y)]\!] = \bigvee_{\tau \in M} [\![\psi(y/\tau)]\!]$.

Examples

Let $\mathcal{M}_L$ be the algebra of Lebesgue measurable subsets of [0; 1] and let Null be the ideal of null sets. The *measure algebra* is MALG $:= \mathcal{M}_L/\mathrm{Null}$.

Then $L^\infty([0;1])$ is a MALG-valued model for the language of rings $\mathcal{L} = \{+, \cdot, 0, 1\}$ where,for $f, g, h \in L^\infty([0;1])$,

$$[\![+(f,g,h)]\!] := \left[\{r \in \mathbb{R} : f(r) + g(r) = h(r)\}\right]_{\mathrm{Null}}.$$

One can prove that $L^\infty([0;1]) \vDash T_{\mathrm{fields}}$:

$$[\![\forall f (f \neq 0 \to \exists g (f \cdot g = 1))]\!] = 1_{\mathrm{MALG}}.$$

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Assume the class of all sets V to be a model of ZFC. Let $M \in V$ a model of (a sufficiently large fragment of) ZFC. Let $\mathsf{B} \in M$ a boolean algebra which M models to be complete. We define in M the *class of* B*-names* M^{B} by induction on Ord^M:

- $M_0^{\mathsf{B}} := \emptyset$, $M_{\alpha+1}^{\mathsf{B}} := \{f : X \to \mathsf{B} : X \subseteq M_\alpha^{\mathsf{B}}\}$;
- $M_\alpha^{\mathsf{B}} := \bigcup_{\beta<\alpha} M_\beta^{\mathsf{B}}$ if α is a limit ordinal;
- $M^{\mathsf{B}} := \bigcup_{\alpha\in\mathrm{Ord}^M} M_\alpha^{\mathsf{B}}$.

The boolean value of $=$, $\in$ and $\subseteq$ in M^{B} is:

$$[\![x \in y]\!] := \bigvee_{t\in\mathrm{dom}(y)} ([\![x = t]\!] \wedge y(t));$$

$$[\![x \subseteq y]\!] := \bigwedge_{t\in\mathrm{dom}(x)} (x(t) \to [\![t \in y]\!]);$$

$$[\![x = y]\!] := [\![x \subseteq y]\!] \wedge [\![y \subseteq x]\!].$$

Quotients of B-valued models

Let $\mathcal{M}$ a B-valued model for $\mathcal{L}$, and F a filter over B. Consider the equivalence relation

$$\tau \equiv_F \sigma \qquad \Longleftrightarrow \qquad [\![\tau = \sigma]\!] \in F.$$

The B$/F$-valued model $\mathcal{M}/F = \langle M/F, R_i^{\mathcal{M}/F} : i \in I, c_j^{\mathcal{M}/F} : j \in J \rangle$ is defined letting:

- $M/F := M/\equiv_F$;
- for any n-ary relation symbol R in $\mathcal{L}$

$$R^{\mathcal{M}/F}([\tau_1]_F, \ldots, [\tau_n]_F) = [[\![R(\tau_1, \ldots, \tau_n)]\!]]_F \in \mathrm{B}/F;$$

- For any constant symbol c in $\mathcal{L}$, $c^{\mathcal{M}/F} = [c^{\mathcal{M}}]_F$.

In particular, if G is an ultrafilter, $\mathcal{M}/G$ is a traditional first order structure.

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Fullness

We will assume B to be complete.

Definition

Given a first order signature $\mathcal{L}$, a B-valued model $\mathcal{M}$ for $\mathcal{L}$ is *full* if for all ultrafilters G on B, all $\mathcal{L}$-formulae $\phi(x_1, \dots, x_n)$ and all $\tau_1, \dots, \tau_n \in \mathcal{M}$

$$\mathcal{M}/_G \models \phi([\tau_1]_G, \dots, [\tau_n]_G) \quad \text{if and only if} \quad [\![\phi(\tau_1, \dots, \tau_n)]\!]^{\mathcal{M}} \in G.$$

The MALG*-valued model* $L^\infty([0;1])$ *is not full for* $\mathcal{L} = \{+, \cdot, 0, 1\}$ *since* $L^\infty([0;1])/G$ *is not a field for any* $G \in \mathrm{St}(\mathrm{B})$.

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Theorem (Łoś Theorem for boolean valued models)

Let $\mathcal{M}$ be a B*-valued model for the signature $\mathcal{L}$.*
The following are equivalent:

1. $\mathcal{M}$ *is full, i.e.* $\mathcal{M}/_G \models \phi([\tau_1]_G, \dots, [\tau_n]_G) \iff [\![\phi(\tau_1, \dots, \tau_n)]\!]^{\mathcal{M}} \in G$;
2. *for all* $\mathcal{L}_{\mathcal{M}}$*-formulae* $\phi(x_0, \dots, x_n)$ *and all* $\tau_1, \dots, \tau_n \in \mathcal{M}$ *there exists* $\sigma_1, \dots, \sigma_m \in \mathcal{M}$ *such that*

$$\bigvee_{\sigma \in \mathcal{M}} [\![\phi(\sigma, \tau_1, \dots, \tau_n)]\!] = \bigvee_{i=1}^{m} [\![\phi(\sigma_i, \tau_1, \dots, \tau_n)]\!]$$

Mixing property

Definition

A B-valued model $\mathcal{M}$ satisfies the *mixing property* if for every antichain $A \subset \mathrm{B}$, and for every subset $\{\tau_a : a \in A\} \subseteq M$, there exists $\tau \in M$ such that

$$a \leq [\![\tau = \tau_a]\!] \text{ for every } a \in A.$$

Proposition

Let $\mathcal{M}$ be a B*-model for $\mathcal{L}$ satisfying the mixing property. Then $\mathcal{M}$ is full.*

If M is a countable model of ZFC, then M^{B} is full but not mixing.

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Presheaves and sheaves

For $(P,\leq)$ partial order, a *P-presheaf* is a contravariant functor $P \to \mathrm{Set}$.

Assume $(P,\leq)$ is also upward complete. A P-presheaf $\mathcal{F}$ is a *P- sheaf* if for every family $\{p_i : i \in I\} \subseteq P$ with $p := \bigvee_P \{p_i : i \in I\}$:

1. if $f, g \in \mathcal{F}(p)$ are such that
$$\mathcal{F}(p_i \leq p)(f) = \mathcal{F}(p_i \leq p)(g) \quad \textit{for every } i \in I,$$
then $f = g$;
2. if $\{f_i \in \mathcal{F}(p_i) : i \in I\}$ is a *matching family* i.e. such that, for $i \neq j$ and $q \leq p_i, p_j$,
$$\mathcal{F}(q \leq p_i)(f_i) = \mathcal{F}(q \leq p_j)(f_j),$$
then there exists a *collation* $f \in \mathcal{F}(p)$ such that
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Boolean valued models as presheaves

For every $b \in \mathsf{B}^+$ let F_b to be the filter generated by b.

Given a complete boolean algebra B *and a* B*-valued model* $\mathcal{M}$*, its associated presheaf* $\mathcal{F}_{\mathcal{M}} : (\mathsf{B}^+)^{op} \to \mathsf{Set}$ *is such that*

- $\mathcal{F}_{\mathcal{M}}(b) = \mathcal{M}/_{F_b}$ *for any* $b \in \mathsf{B}^+$*;*
- $\mathcal{F}_{\mathcal{M}}(b \leq c)$ *is the map*

$$i^{\mathcal{M}}_{bc} : \mathcal{M}/_{F_c} \to \mathcal{M}/_{F_b}$$
$$[\tau]_{F_c} \mapsto [\tau]_{F_b}.$$

Theorem (Monro - '86)

Let B *be a complete boolean algebra. Then the* B*-valued model* $\mathcal{M}$ *has the mixing property if and only if the presheaf* $\mathcal{F}_{\mathcal{M}}$ *is a sheaf.*

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Bundles

Definition

A *bundle* over X is a continuous map $p : E \to X$.

A *section* of the bundle $p : E \to X$ is a continuous map $s : X \to E$ such that $p \circ s$ is the identity of X. If s is defined only on a open subset $U \subset X$, it is called a *local section*.

Boolean valued models as bundles

Consider, for a B-valued model $\mathcal{M}$, the set

$$E_{\mathcal{M}} := \bigsqcup_{G \in \mathrm{St}(\mathsf{B})} \mathcal{M}/G = \{[\sigma]_G : \sigma \in M, G \in \mathrm{St}(\mathsf{B})\}$$

and $p : E_{\mathcal{M}} \to \mathrm{St}(\mathsf{B})$ such that $[\sigma]_G \mapsto G$.

For every $\sigma \in \mathcal{M}$ define a global section $\dot{\sigma} : \mathrm{St}(\mathsf{B}) \to E_{\mathcal{M}}$ as $\dot{\sigma}(G) := [\sigma]_G$.

Define a topology on $E_{\mathcal{M}}$ by taking as a base the family

$$\mathcal{B} := \{\dot{\sigma}[N_b] = \{[\sigma]_G : b \in G\} : \sigma \in M, b \in \mathsf{B}\}.$$

A local section $s : U \to E_{\mathcal{M}}$ from some open subset $U \subseteq \mathrm{St}(\mathsf{B})$ is *induced* by some element $\sigma \in \mathcal{M}$ if $s = \dot{\sigma} \restriction U$.

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Mixing models have trivial global sections

Theorem

Let B *a complete boolean algebra. For a* B*-valued model* $\mathcal{M}$ *the following are equivalent:*

1. $\mathcal{M}$ *has the mixing property;*
2. *every local section of the bundle* $E_{\mathcal{M}} \to \mathrm{St}(\mathsf{B})$ *can be extended to a global section induced by an element of* $\mathcal{M}$*;*
3. $\mathcal{F}_{\mathcal{M}}$ *is isomorphic to the sheaf of continuous sections of* $E_{\mathcal{M}} \to \mathrm{St}(\mathsf{B})$.

..and for the fullness property?

For every formula $\phi(x)$ in the language define a bundle

$$E^{\phi}_{\mathcal{M}} = \{[\sigma]_G : [\![\phi(\sigma)]\!] \in G \in \mathrm{St}(\mathsf{B})\}.$$

over $N_{[\![\exists x \phi(x)]\!]}$ with the map $p_\phi : [\sigma]_G \mapsto G$.

Theorem

For a B*-valued model* $\mathcal{M}$ *for the language* $\mathcal{L}$ *the following are equivalent:*

- $\mathcal{M}$ *is full;*
- *for every formula* $\phi(x)$*,* $p_\phi : E^{\phi}_{\mathcal{M}} \to N_{[\![\exists x \phi(x)]\!]}$ *is surjective;*
- *for every formula* $\phi(x)$*,* $p_\phi : E^{\phi}_{\mathcal{M}} \to N_{[\![\exists x \phi(x)]\!]}$ *has at least one global section.*

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