Some Results Related to Ordinal Ramsey Theory

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Definition

$\alpha \longrightarrow (\beta, \gamma)^2$ means that every graph on a set of size α has an independent set of size β or a complete subgraph of size γ.

Definition

$r(\beta, \gamma) = \alpha$ means $\alpha \longrightarrow (\beta, \gamma)^2$ but $\delta \not\longrightarrow (\beta, \gamma)^2$ for all $\delta < \alpha$.

Example

$r(3, 3) = 6.$

Notation

For a graph G let

- $n = n_G$ *be the number of its vertices,*
- $e = e_G$ *be the number of its edges and*
- $d = d_G = \frac{2e_G}{n_G}$ *be its average degree.*
- $d^{\mathit{max}} = d_G^{\mathit{max}}$ *be its maximum degree.*
- $\alpha = \alpha_G$ *the minimal size of an independent set.*

Theorem (Turán, ?)

$$\alpha \geqslant \frac{n}{d+1}.$$

Observation

For triangle-free graphs, $\alpha \geqslant d$.

Corollary

$n(n+1) \longrightarrow (n,3)^2$.

Theorem (Erdős, 1961)

There is a constant $c > 0$ *such that* $\left\lfloor \frac{cn^2}{(\ln(n))^2} \right\rfloor \not\longrightarrow (n,3)^2$ *for all natural numbers* n.

Theorem (Graver & Yackel, 1968)

There is a constant $c > 0$ *such that* $\left\lfloor \frac{cn^2 \ln(\ln(n))}{\ln(n)} \right\rfloor \longrightarrow (n,3)^2$ *for all natural numbers* n.

Theorem (Ajtai, Komlós & Szemerédi, 1980)

There is a constant $c > 0$ *such that* $\left\lfloor \frac{cn^2}{\ln(n)} \right\rfloor \longrightarrow (n, 3)^2$ *for all* $n \in \omega \setminus 2$.

Theorem (Shearer, 1982)

$\alpha \geqslant \frac{n(d\ln(d) + 1 - d)}{(d-1)^2}$ *for triangle-free graphs.*

Corollary

An version of the Theorem of Ajtai, Komlós, and Szemerédi with smaller c.

Theorem (Kim, 1995)

There is a constant $c > 0$ *such that* $\left\lfloor \frac{cn^2}{\ln(n)} \right\rfloor \not\longrightarrow (n, 3)^2$ *for all* $n \in \omega \setminus 2$.

Corollary

There is a constant $c > 0$ *such that* $\left\lfloor \frac{cn^2}{\ln(n)} \right\rfloor \not\longrightarrow (I_n, L_3)^2$ *for all* $n \in \omega \setminus 2$.

Notation

$k \longrightarrow (I_m, L_n)^2$ *if and only if every oriented graph on a set of size* k *has an independent set of size* m *or a complete cyclefree subgraph of size* n.

Theorem (Erdős & Rado for $\kappa = \omega$, Baumgartner for cardinals $\kappa > \omega$)

$\kappa k \longrightarrow (\kappa m, n)^2$ *if and only if* $k \longrightarrow (I_m, L_n)^2$ *for all infinite cardinals* κ.

Theorem (Ramsey's Theorem for two colours)

$\omega \longrightarrow (\omega, \omega)^n$ *for every natural number* n.

Definition

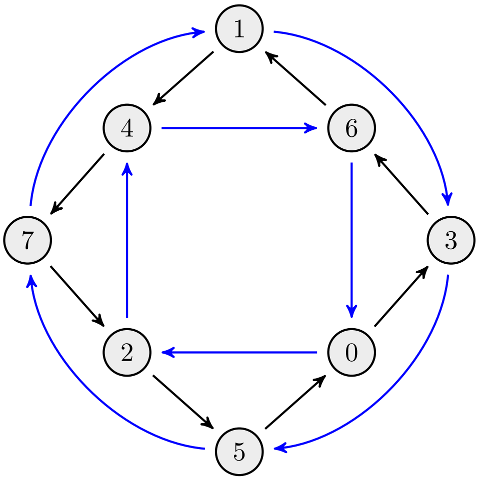
$r(I_k, L_m) = n$ means $n \longrightarrow (I_k, L_m)^2$ but $p \not\longrightarrow (I_k, L_m)^2$ for all $p < n$.

Example (Erdős & Rado, 1956)

$r(I_2, L_3) = 4$.

Example (Bermond, 1974)

$8 \not\longrightarrow (I_3, L_3)^2$.



$$x \mapsto x + 3$$
$$x \mapsto x + 2.$$

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Example (Larson & Mitchell, 1997)

$13 \not\longrightarrow (I_4, L_3)^2.$

Theorem (Larson & Mitchell, 1997)

$n^2 \longrightarrow (I_n, L_3)^2.$

Theorem (Ihringer, Rajendraprasad & W.)

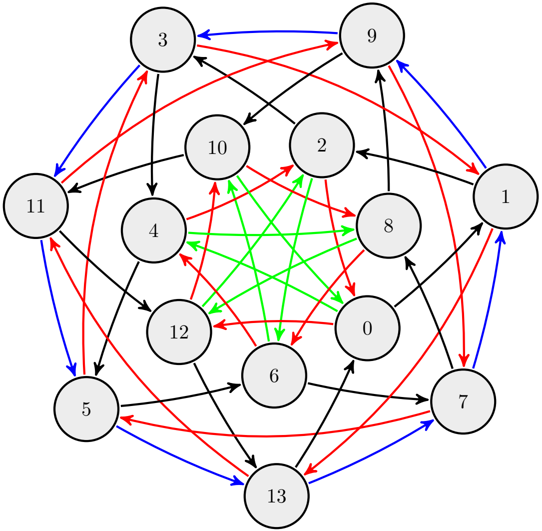
$n^2 - n + 3 \longrightarrow (I_n, L_3)^2$ *for* $n \in \omega \setminus 2$.

Example (Rajendraprasad)

$14 \not\longrightarrow (I_4, L_3)^2.$

Corollary

$r(I_4, L_3) = 15.$



$x \mapsto x + 1$

$x \mapsto x - 2$

$x \mapsto x{+}4$ if x is even

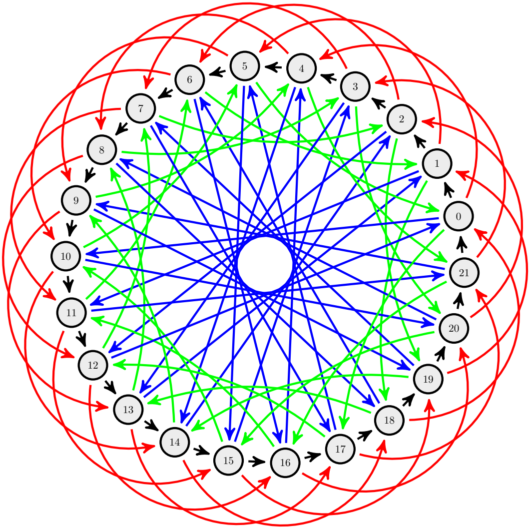
$x \mapsto x{-}6$ if x is odd.

Example (Rajendraprasad)

$22 \not\rightarrow (I_5, L_3)^2$.

Corollary

$r(I_4, L_3) = 23$.



$x \mapsto x + 1$

$x \mapsto x + 4$

$x \mapsto x - 5$

$x \mapsto x + 10$

Theorem (Alon, 1996)

Considering a graph with at least one edge in which the neighbourhood of any vertex is r-colourable, we have

$$\alpha \geqslant \frac{n\,\mathrm{ld}(d^{\mathsf{max}})}{160 d^{\mathsf{max}}\,\mathrm{ld}(r+1)}.$$

Corollary

$$\left\lfloor \frac{508n^2}{\mathrm{ld}(n)} \right\rfloor \longrightarrow (I_n, L_3)^2.$$

Lemma (Alon, 1996)

Let $\mathcal{F}$ be a family of k distinct subsets of an n-element set X . Then the average size of a member of $\mathcal{F}$ is at least

$$\frac{\mathrm{ld}(k)}{10\,\mathrm{ld}\left(\frac{\mathrm{ld}(k)+n}{\mathrm{ld}(k)}\right)}.$$

Lemma (Tentative Improvement, Almost Proven)

Let $\mathcal{F}$ be a family of k distinct subsets of an n-element set X . Then the average size of a member of $\mathcal{F}$ is at least

$$\frac{(3-\sqrt{8})\,\mathrm{ld}(k)}{\mathrm{ld}\left(\frac{\mathrm{ld}(k)+n}{\mathrm{ld}(k)}\right)}.$$

Note that $3-\sqrt{8} > \frac{1}{6}$.

The Lemma would yield the following:

Proposition (Almost Proven)

Considering a graph with at least one edge in which the neighbourhood of any vertex is 2-colourable, we have $\alpha \geqslant \frac{n\,\mathrm{ld}(d^{\mathsf{max}})}{13d^{\mathsf{max}}}$.

Corollary (Almost Proven)

$\left\lfloor \frac{26n^2}{\mathrm{ld}(n)} \right\rfloor \longrightarrow (I_n, L_3)^2$ *for all natural numbers* n.

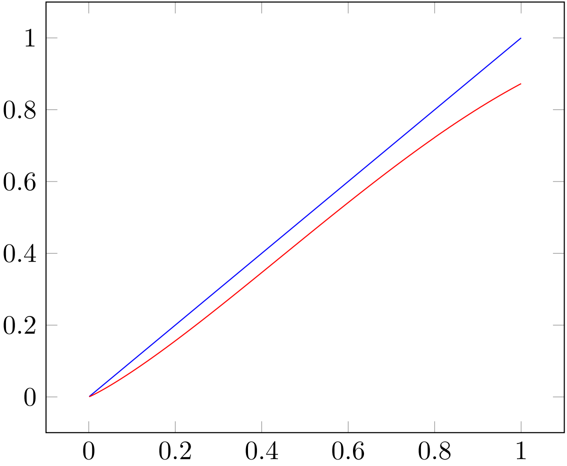
This all hinges on proving the seemingly true inequality

$$H\left(\frac{(2-\sqrt{2})\,x}{2\,\mathrm{ld}\left(1+\frac{1}{x}\right)}\right) \leqslant x \text{ for all } x \in [0,1]$$

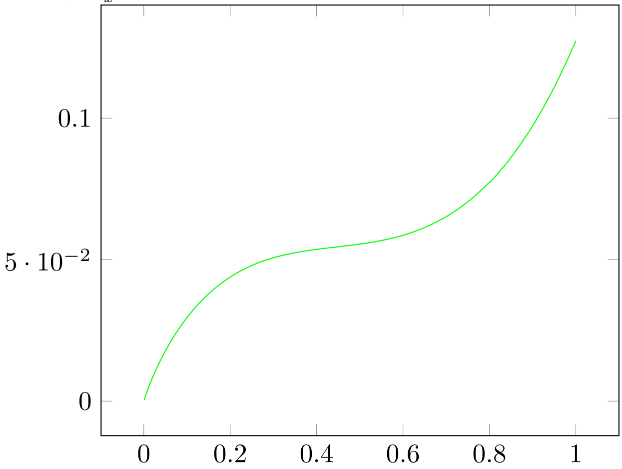
where H is the binary entropy function

$$\begin{aligned} H :]0,1[&\longrightarrow \mathbb{R} \\ x &\longmapsto -\,\mathrm{ld}\,(x)\,x - \mathrm{ld}\,(1-x)\,(1-x) \end{aligned}$$

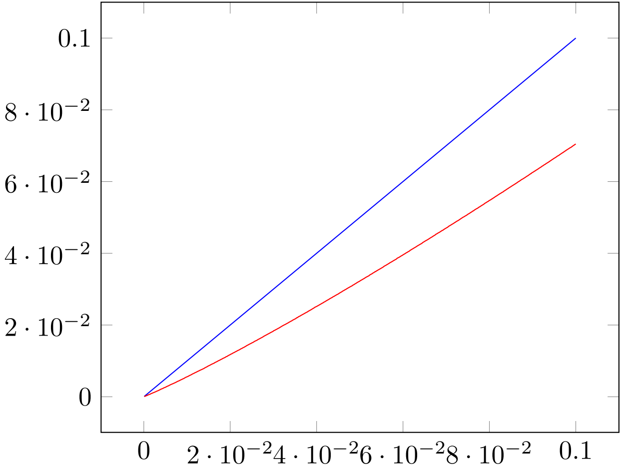
$H(\frac{(2-\sqrt{2})x}{2\,\mathrm{ld}(1+\frac{1}{x})})$ and x



$H(\frac{(2-\sqrt{2})x}{2\,\mathrm{ld}(1+\frac{1}{x})}) - x$



$H(\frac{(2-\sqrt{2})x}{2\,\mathrm{ld}(1+\frac{1}{x})})$ and x, closer to 0.



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5	14	25						
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Observation

$r(n+1,3) - r(n,3) \leqslant n+1.$

Proof.

Fix a vertex v in a graph on $r(n,3)+n+1$ vertices. Then either v has a neighbourhood of $n+1$ vertices or v is independent from a set of size $r(n,3)$. □

Proposition (Graver & Yackel, 1968)

Let G be a $(3, y)$-graph on n points with e edges. Let p_1 and p_2 be two points of G a distance of at least 5 apart (i.e., any path joining p_1 and p_2 has at least 5 edges). Denote the valence of p_i by $v_i (i = 1, 2)$; and let K_i represent the v_i points which are adjacent to p_i. Finally let G' be the graph formed by removing from G the points p_1 and p_2 and all edges with p_1 or p_2 as end-points, and then adding all edges between points in K_1 and points in K_2. Then G' is a $(3, y-1)$-graph on $(n-2)$ points with $[e + (v_1 - 1)(v_2 - 1) - 1]$ edges

Corollary

$r(n+1, 3) - r(n, 3) \geqslant 3$ *for all* $n \in \omega \setminus 2$.

Conjecture (Erdős & Sós)

$$\liminf_{n \nearrow \infty} r(n+1, 3) - r(n, 3) = \infty.$$

$r(I_m, L_n)$.

	2	3	4	5	m
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4	8	?			
5	14				
6	28				
n					

Observation

$r(I_{n+1}, L_3) - r(I_n, L_3) \leqslant 2n + 1.$

Proof.

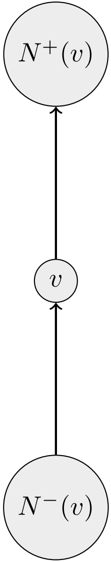
Fix a vertex v in a graph on $r(I_n, L_3) + 2n + 1$ vertices. Then either v has an in-neighbourhood of $n + 1$ vertices or an out-neighbourhood of $n + 1$ vertices or v is independent from a set of size $r(I_n, L_3)$. □

Proposition (W., 2021)

Let e, i, and n be natural numbers. If there is an oriented graph all whose triangles are cyclic and all whose independent sets are smaller than i,with e edges on n vertices one of which is v having degree d, then there is an oriented graph on $n + 5$ vertices with $2d + e + 9$ edges all whose triangles are cyclic and all whose independent sets have size at most i.

Corollary

$r(I_{n+1}, L_3) \geqslant r(I_n, L_3) + 5$ *for all* $n \in \omega \setminus 2$.



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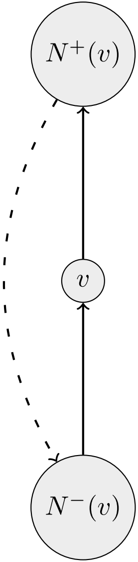

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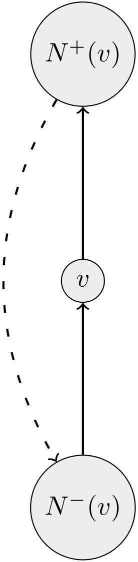

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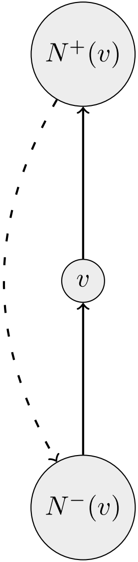
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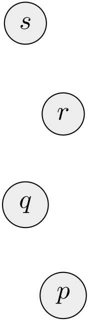
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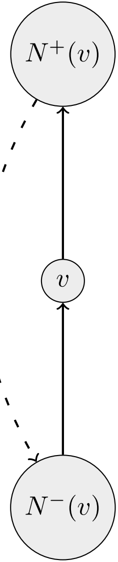

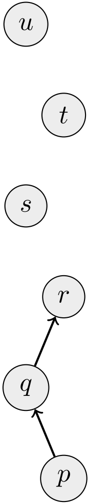

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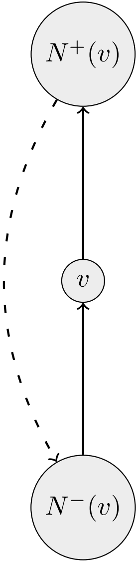

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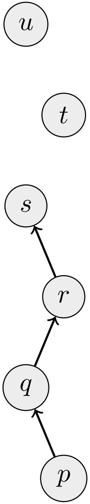

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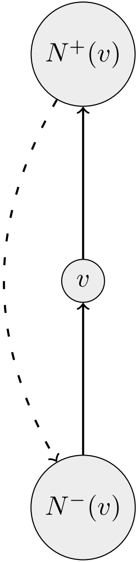
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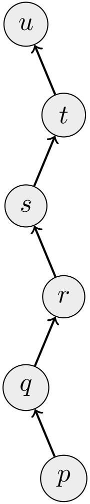
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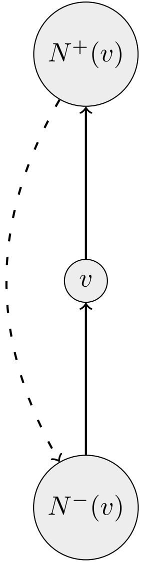

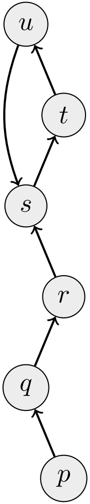

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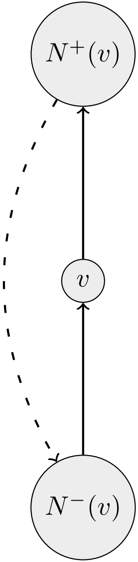
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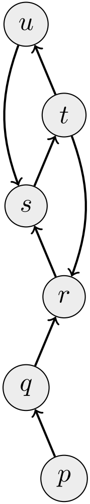
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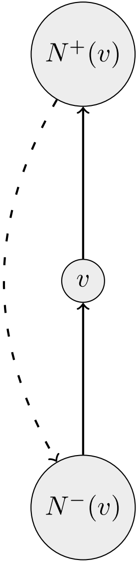
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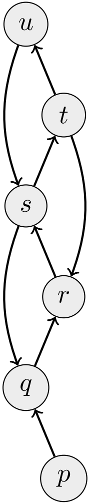
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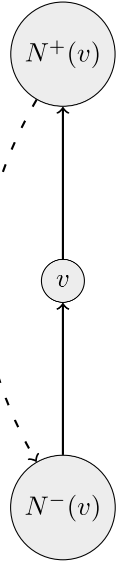
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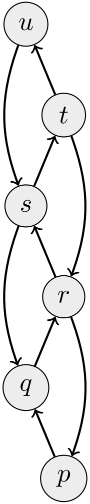
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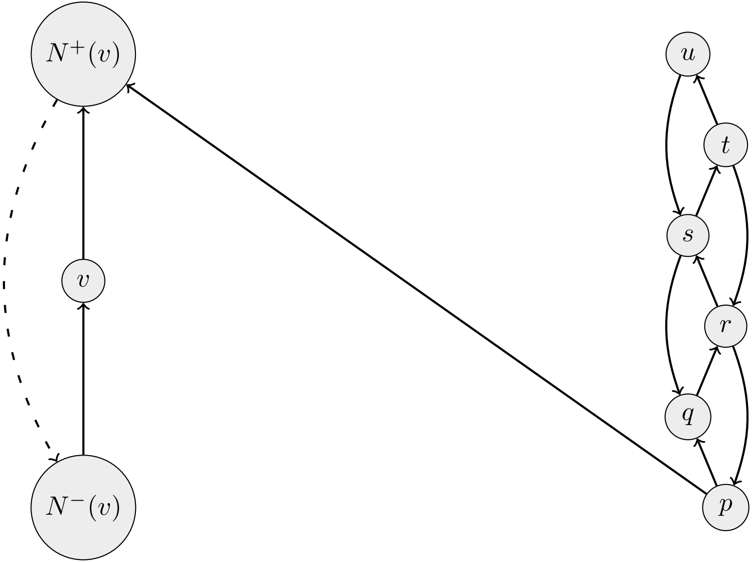

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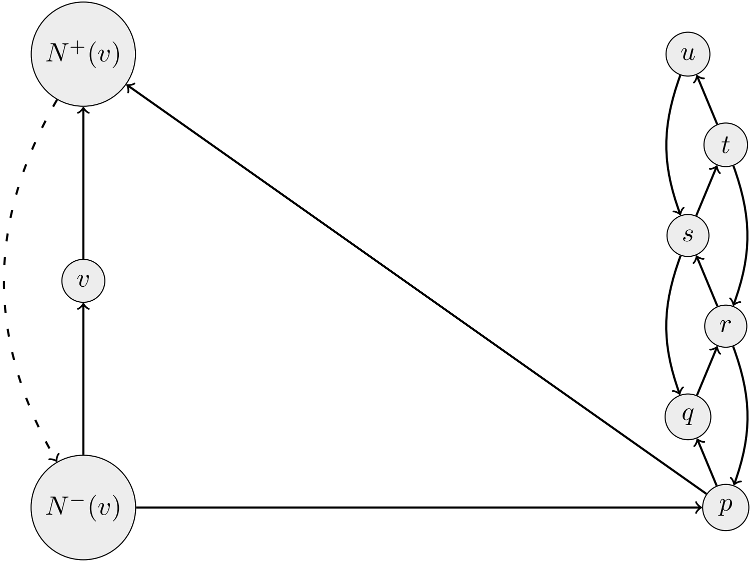

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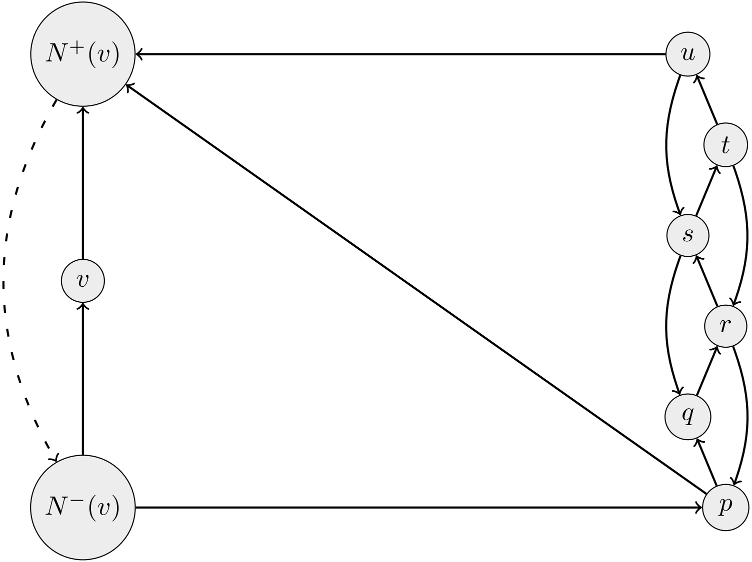
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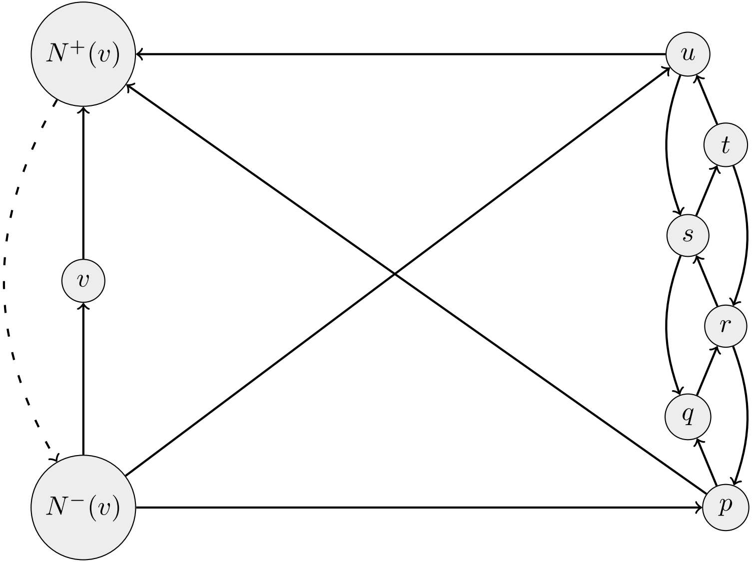
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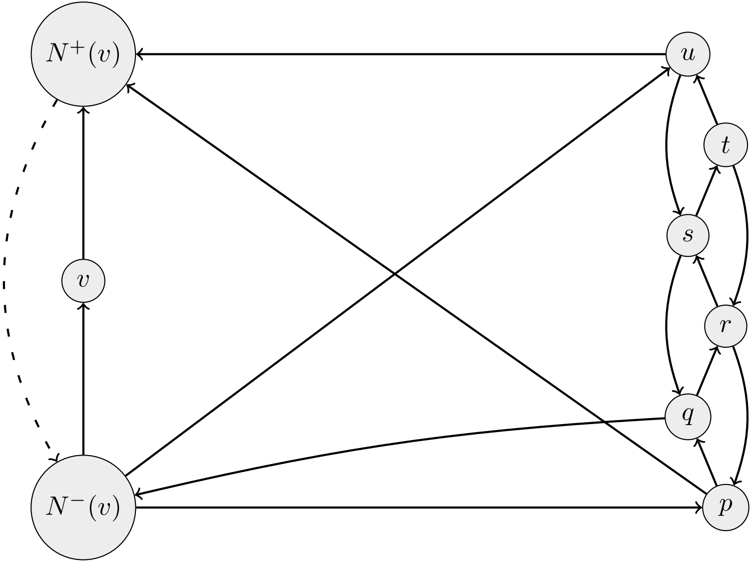
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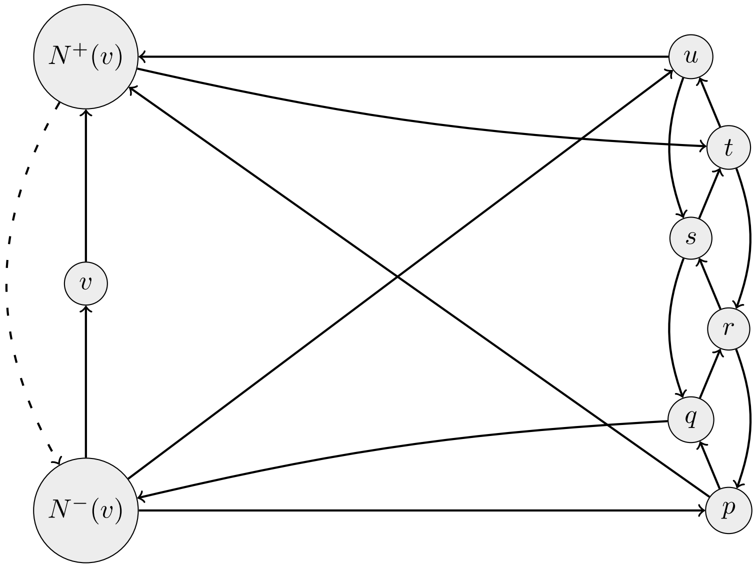
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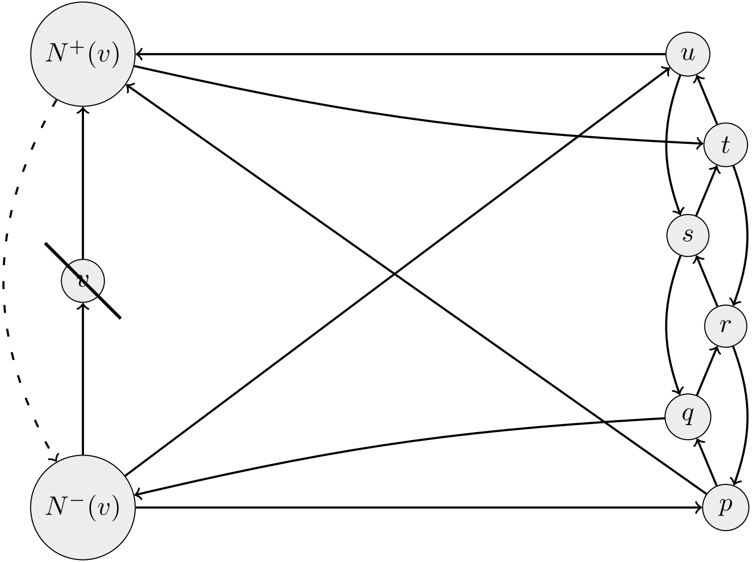

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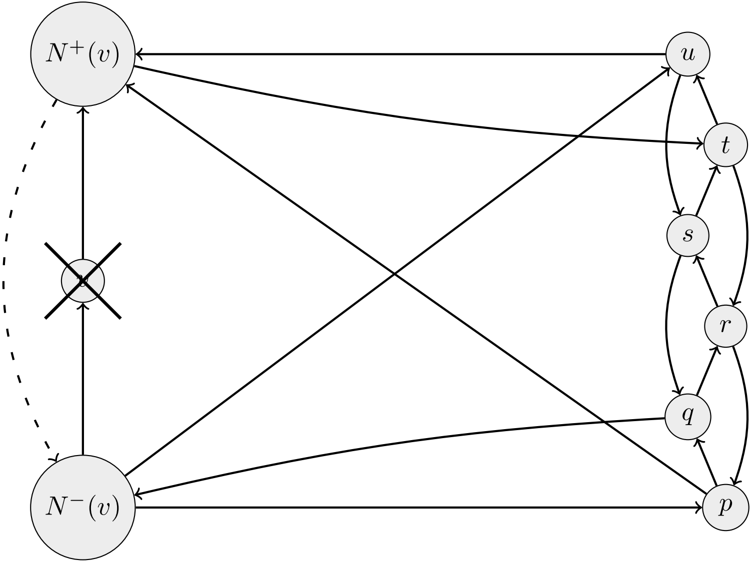
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Question

What is $r(I_3, L_4)^2$?

We know that $r(I_3, L_4) \in 25 \setminus 21 = \{21, 22, 23, 24\}$.
For context:

Theorem (Codish, Frank, Itzhakov & Miller, 2016)

$r(3, 3, 4) = 30$.

Question

$$\liminf_{n \nearrow \infty} r(I_{n+1}, L_3) - r(I_n, L_3) = \infty?$$

Miklós Ajtai, János Komlós, and Endre Szemerédi. A note on Ramsey numbers. *J. Combin. Theory Ser. A*, 29(3):354–360, 1980. ISSN 0097-3165. doi:10.1016/0097-3165(80)90030-8. URL https://doi.org/10.1016/0097-3165(80)90030-8.

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