PARTITIONS AND P-LIKE IDEALS

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Basic notions

A family $\mathcal{I} \subseteq \mathcal{P}(M)$ of subsets of a given (usually countable) set M is called an **ideal on M**, if

- $[M]^{<\omega} \subseteq \mathcal{I}$, $M \notin \mathcal{I}$,
- $\mathcal{I}$ is closed under taking subsets and finite unions.

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- We denote an ideal of all finite subsets of M by Fin.

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- a **P($\mathcal{J}$)-ideal**, if for each countable family $\{I_n : n \in \omega\} \subseteq \mathcal{I}$ there is an $I \in \mathcal{I}$ such that $I_n \subseteq^{\mathcal{J}} I$ for every $n \in \omega$ (where $I_n \subseteq^{\mathcal{J}} I$ iff $I_n \setminus I \in \mathcal{J}$).

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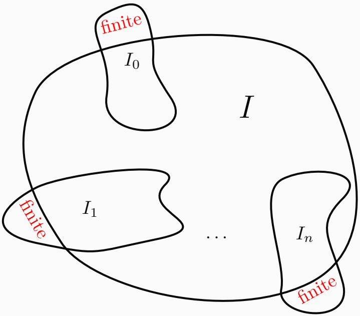
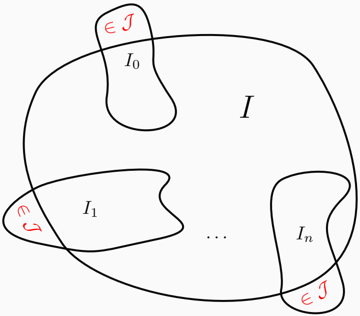
  - P-ideal is just a P(Fin)-ideal

Central Notion



$\mathcal{I}$ is a P-ideal



$\mathcal{I}$ is a P($\mathcal{J}$)-ideal

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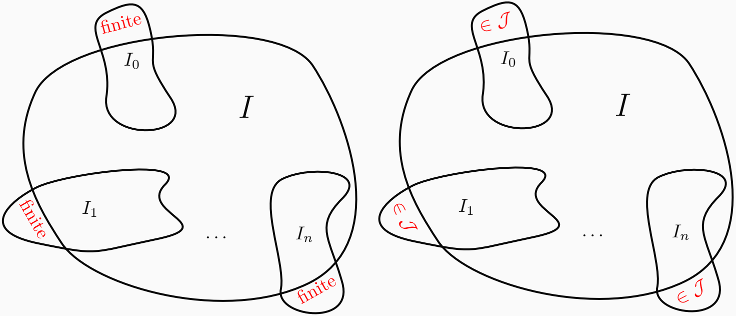


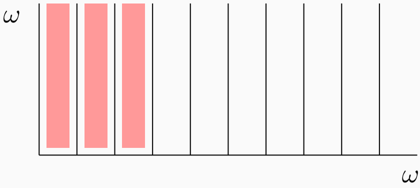
$\mathcal{I}$ is a P-ideal | $\mathcal{I}$ is a P($\mathcal{J}$)-ideal

- note that $\mathcal{I}$ is a P($\mathcal{J}$)-ideal if and only if $\mathfrak{b}(\mathcal{I}, \subseteq^{\mathcal{J}}) \geq \omega_1$

Example

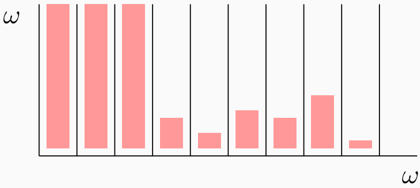
$$A \in \mathrm{Fin} \times \emptyset$$
$$\Updownarrow$$
$$\{n : \{m : (n,m) \in A\} \neq \emptyset\} \in \mathrm{Fin}$$

Fubini product $\mathrm{Fin} \times \emptyset$



$$A \in \mathrm{Fin} \times \mathrm{Fin}$$
$$\Updownarrow$$
$$\{n : \{m : (n,m) \in A\} \notin \mathrm{Fin}\} \in \mathrm{Fin}$$

Fubini product $\mathrm{Fin} \times \mathrm{Fin}$



Example

$\mathbf{Fin} \times \mathbf{Fin}$

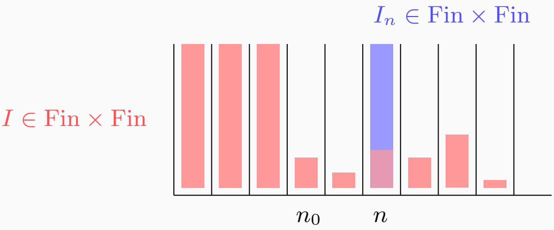
- is not a P-ideal

Example

Fin × Fin

- is not a P-ideal

  consider the sequence of columns $I_n = \{n\} \times \omega$



  for any $I \in \mathrm{Fin} \times \mathrm{Fin}$ there is n_0 such that $I \cap I_n$ is finite for $n > n_0$

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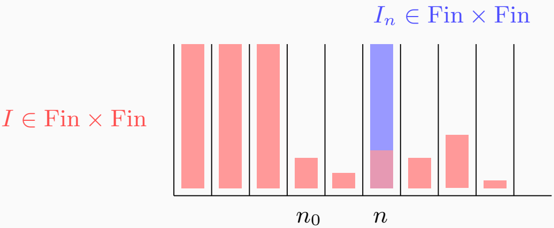

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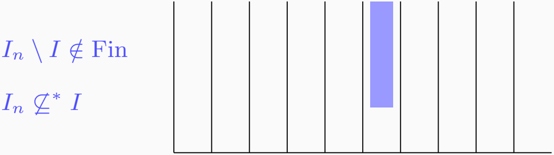


Example

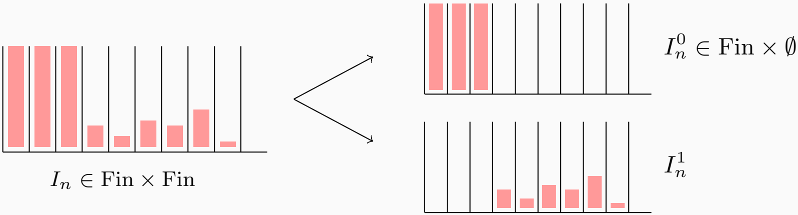
$\mathbf{Fin} \times \mathbf{Fin}$

- is a $\mathrm{P}(\mathrm{Fin} \times \emptyset)$-ideal

Example

Fin × Fin

- <u>is</u> a P(Fin × ∅)-ideal



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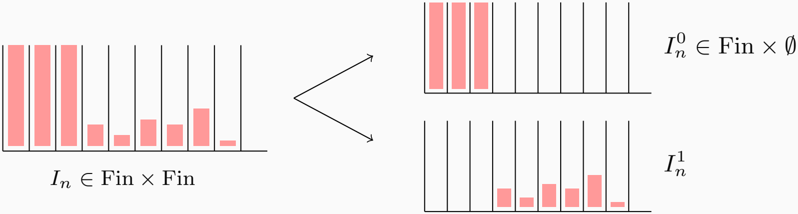

I_n^1 can be covered by an area below graph of function $f_n \in {}^\omega\omega$ for each n

$\rightarrow$ take an area below a function that $\leq^*$-dominates f_n for every n

Motivation

Theorem (M. Mačaj – M. Sleziak [4], 2010)

Let X be a non-discrete first countable topological space and let $\mathcal{I}, \mathcal{J}$ be ideals on ω. The following are equivalent:

1) *$\mathcal{I}$ is a* $\mathrm{P}(\mathcal{J})$*-ideal.*
2) *In the Boolean algebra $\mathcal{P}(\omega)/\mathcal{J}$ the ideal $\mathcal{I}$ corresponds to a σ-directed subset*[1]*.*
3) *For any sequence $\langle x_n : n \in \omega \rangle$ in X, if $\langle x_n : n \in \omega \rangle$ is $\mathcal{I}$-convergent to x then $\langle x_n : n \in \omega \rangle$ is $\mathcal{I}^{\mathcal{J}}$-convergent to x.*

[1] i.e., it contains an upper bound of each countable subset.

Theorem (R. Filipów – M. Staniszewski [1], 2014)

Let X be a non-empty set and $\mathcal{I}, \mathcal{J}$ be ideals on ω. The following are equivalent:

1) $\mathcal{I}$ *is a* $\mathrm{P}(\mathcal{J})$*-ideal.*
2) *For any sequence* $\langle f_n : n \in \omega \rangle$ *of real-valued functions on* X*, if* $\langle f_n : n \in \omega \rangle$ *is* $\mathcal{I}$*-uniformly convergent to* f *then* $\langle f_n : n \in \omega \rangle$ *is* $(\mathcal{I}, \mathcal{J})$*-equally convergent to* f*.*

Basic properties

Lemma (R. Filipów – M. Staniszewski)

a) *$\mathcal{I}$ is a* $\mathrm{P}(\mathcal{I})$*-ideal for every* $\mathcal{I}$*.*

b) *If* $\mathcal{I}$ *is* $\mathrm{P}(\mathcal{J})$ *and* $\mathcal{J}' \supseteq \mathcal{J}$*, then* $\mathcal{I}$ *is* $\mathrm{P}(\mathcal{J}')$*.*

c) *If* $\mathcal{I}, \mathcal{J}$ *are maximal then* $\mathcal{I}$ *is* $\mathrm{P}(\mathcal{J})$ *and* $\mathcal{J}$ *is* $\mathrm{P}(\mathcal{I})$*.*

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c) *If* $\mathcal{I}, \mathcal{J}$ *are maximal then* $\mathcal{I}$ *is* $\mathrm{P}(\mathcal{J})$ *and* $\mathcal{J}$ *is* $\mathrm{P}(\mathcal{I})$.

- In particular, P-ideals are $P(\mathcal{J})$-ideals for every $\mathcal{J}$.

Basic properties

Proposition

If $\mathcal{I}_{\mathrm{m}}$ *is a maximal ideal on* M*, then* $\mathcal{J}$ *is a* $\mathrm{P}(\mathcal{I}_{\mathrm{m}})$*-ideal for every ideal* $\mathcal{J}$ *on* M*.*

Basic properties

Proposition

If $\mathcal{I}_{\mathrm{m}}$ is a maximal ideal on M, then $\mathcal{J}$ is a $\mathrm{P}(\mathcal{I}_{\mathrm{m}})$-ideal for every ideal $\mathcal{J}$ on M.

There exist tall non-$\mathrm{P}(\mathcal{J})$-ideals for a broad class of ideals $\mathcal{J}$.

Proposition

If $\mathcal{J}$ is ideal on a countable set M such that there is $\mathcal{X} = \{X_n : n \in \omega\} \subseteq \mathcal{J}^+$ of pairwise disjoint sets, then there is a tall ideal $\mathcal{I}$ such that $\mathcal{I}$ is not a $\mathrm{P}(\mathcal{J})$-ideal.

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- this includes non-tall ideals, ideals generated by MAD families, meager ideals, asymptotic density zero ideal, van der Waerden ideal etc.

Inducing partitions

$\mathrm{P}(\mathcal{J})$	$\mathcal{J} = \mathrm{Fin}$	$\mathcal{J} = \mathrm{Fin} \times \emptyset$	$\mathcal{J} = \mathrm{Fin} \times \mathrm{Fin}$
Fin	✓	✓	✓
$\mathrm{Fin} \times \emptyset$	✗	✓	✓
$\mathrm{Fin} \times \mathrm{Fin}$	✗	✓	✓

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What about isomorphic copies of these ideals?

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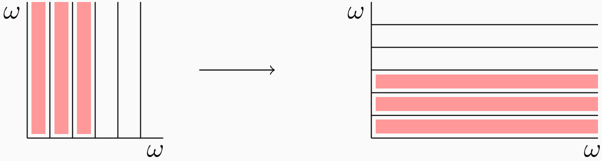
What about isomorphic copies of these ideals?

Consider e. g. a bijection $f\colon \omega \times \omega \to \omega \times \omega$ defined by $f(n, m) = (m, n)$ and $f(\mathrm{Fin} \times \emptyset) = \{f[I] : I \in \mathrm{Fin} \times \emptyset\}$.



$I \in \mathrm{Fin} \times \emptyset$ $\qquad$ $f[I] \in f(\mathrm{Fin} \times \emptyset)$

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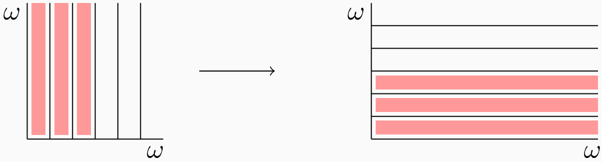

$I \in \mathrm{Fin} \times \emptyset$ $\qquad$ $f[I] \in f(\mathrm{Fin} \times \emptyset)$

$f(\mathrm{Fin} \times \emptyset)$ is just an ideal generated by $\mathcal{A} = \{\omega \times \{n\} : n \in \omega\}$

Inducing partitions

Observation: Every isomorphic copy of the ideals Fin, $\mathrm{Fin} \times \emptyset$, $\mathrm{Fin} \times \mathrm{Fin}$ can be expressed solely in terms of infinite partitions of $\omega \times \omega$ into infinite sets.

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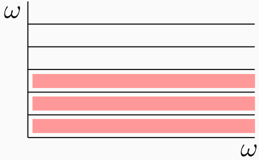
E. g. let $\mathcal{A} = \{\omega \times \{n\} : n \in \omega\}$



$I \in (\mathrm{Fin} \times \emptyset)(\mathcal{A})$

Relations $\subseteq^{\restriction}$ and $\perp$

$P(\mathcal{J})$	$\mathcal{J} = \mathrm{Fin}$	$\mathcal{J} = \mathrm{Fin} \times \emptyset$	$\mathcal{J} = \mathrm{Fin} \times \mathrm{Fin}$
Fin	?	?	?
$(\mathrm{Fin} \times \emptyset)(\mathcal{A})$	?	?	?
$(\mathrm{Fin} \times \mathrm{Fin})(\mathcal{A})$	?	?	?

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Are there some general rules describing the relationships?

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- If $\mathcal{I} \subseteq^{\restriction} \mathcal{J}$ then $\mathcal{I}$ is a $\mathrm{P}(\mathcal{J})$-ideal. The converse is not true in general.

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- note that $\mathcal{I} \subseteq^{\restriction} \mathcal{J}$ is in fact equivalent to $\mathtt{cof}(\mathcal{I})$-$\mathrm{P}(\mathcal{J}, \mathcal{I})$-ideal using the notation from [5].

Relations $\subseteq^{\restriction}$ and $\perp$

We say that ideals $\mathcal{I}$ and $\mathcal{J}$ are **orthogonal** ($\mathcal{I} \perp \mathcal{J}$ for short) if there is $X \in \mathcal{P}(M)$ such that $X \in \mathcal{I}$ and $M \setminus X \in \mathcal{J}$.

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- If $\mathcal{I} \perp \mathcal{J}$ then $\mathcal{I} \subseteq^{\restriction} \mathcal{J}$, hence $\mathcal{I}$ is a $\mathrm{P}(\mathcal{J})$-ideal.

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- If $\mathcal{I} \perp \mathcal{J}$ then $\mathcal{I} \subseteq^{\restriction} \mathcal{J}$, hence $\mathcal{I}$ is a $\mathrm{P}(\mathcal{J})$-ideal.
- Two distinct maximal ideals are orthogonal.

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- Two distinct maximal ideals are orthogonal.
- For any pair of ideals $\mathcal{I}, \mathcal{J} \neq \mathrm{Fin}$ there is an isomorphism f such that $f(\mathcal{I}) \perp \mathcal{J}$.

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- If $\mathcal{I} \perp \mathcal{J}$ then $\mathcal{I} \subseteq^{\restriction} \mathcal{J}$, hence $\mathcal{I}$ is a $\mathrm{P}(\mathcal{J})$-ideal.
- Two distinct maximal ideals are orthogonal.
- For any pair of ideals $\mathcal{I}, \mathcal{J} \neq \mathrm{Fin}$ there is an isomorphism f such that $f(\mathcal{I}) \perp \mathcal{J}$.
- None of the ideals $\mathrm{Fin}, \mathrm{Fin} \times \emptyset, \mathrm{Fin} \times \mathrm{Fin}$ are orthogonal.

Relations $\subseteq^{\restriction}$ and $\perp$

Proposition

If there is an AD family $\mathcal{C}$ such that $\mathcal{I}$ is generated by $\mathcal{C}$ then the following statements are equivalent.

1) $\mathcal{I}$ *is a* $\mathrm{P}(\mathcal{J})$*-ideal.*
2) $\mathcal{C} \setminus \mathcal{J}$ *is finite.*
3) $\mathcal{I} \subseteq^{\restriction} \mathcal{J}$.

[2] $\mathcal{I}$ is **nowhere tall**, if $(\forall A \in \mathcal{I}^+)(\exists B \in [A]^\omega)\ \mathcal{I} \restriction B = [B]^{<\omega}$ (see e. g. [2]).

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If $\mathcal{C}$ is MAD and $\mathcal{J}$ is nowhere tall[2], then we can add

4) $\mathcal{I} \perp \mathcal{J}$.

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Critical ideals on $\omega \times \omega$

Proposition

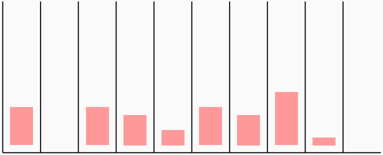
The following statements are equivalent.

1) $(\mathrm{Fin} \times \emptyset)(\mathcal{A})$ *is a* $\mathrm{P}(\mathcal{J})$*-ideal.*
2) $(\mathrm{Fin} \times \mathrm{Fin})(\mathcal{A})$ *is a* $\mathrm{P}(\mathcal{J})$*-ideal.*
3) $\mathcal{A} \setminus \mathcal{J}$ *is finite.*
4) $(\mathrm{Fin} \times \emptyset)(\mathcal{A}) \subseteq^{\restriction} \mathcal{J}$.

Critical ideals on $\omega \times \omega$

$\mathbf{(\emptyset \times Fin)(\mathcal{A})}$ – the family of all sets with finite intersection with each element of $\mathcal{A}$ (P-ideal)

$I \in (\emptyset \times \mathrm{Fin})(\mathcal{A})$

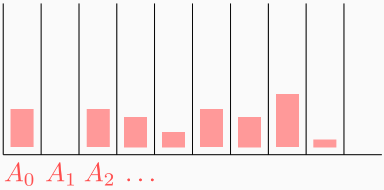


A_0 A_1 A_2 ...

Critical ideals on $\omega \times \omega$

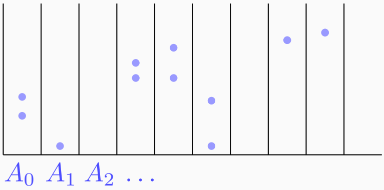
$\mathbf{(\emptyset \times Fin)(\mathcal{A})}$ – the family of all sets with finite intersection with each element of $\mathcal{A}$ (P-ideal)

$I \in (\emptyset \times \mathrm{Fin})(\mathcal{A})$



$\boldsymbol{\mathcal{S}el(\mathcal{A})}$ – the ideal generated by the family of all selectors of $\mathcal{A}$

$I \in \mathcal{S}el(\mathcal{A})$



Critical ideals on $\omega \times \omega$

Let $\mathcal{A}$ be an infinite partition of $\omega \times \omega$ to infinite sets. Set of partial functions $g_0, \ldots, g_{k-1}$ is called a ***k*-tower of monochromatic functions** (with respect to $\mathcal{A}$), if

- there are $A_{i_0}, \ldots, A_{i_{k-1}} \in \mathcal{A}$ such that $g_j \subseteq A_{i_j}$ for $j < k$,
- there is $a \in [\omega]^\omega$ such that $\mathrm{dom}(g_j) = a$ for each $j < k$,
- $g_i \cap g_j = \emptyset$ for $i, j < k$, $i \neq j$.

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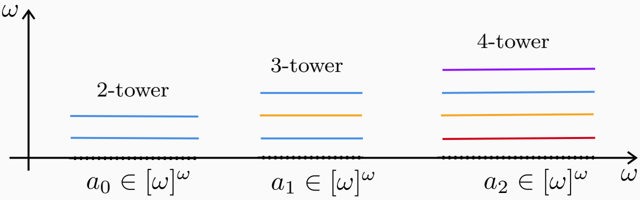
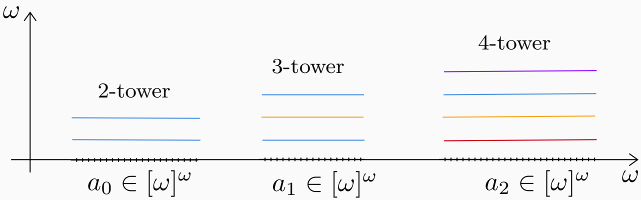
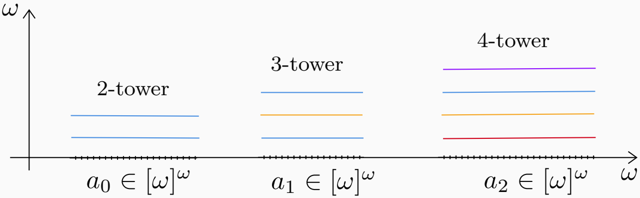


Critical ideals on $\omega \times \omega$



Critical ideals on $\omega \times \omega$



Theorem

The following statements are equivalent.

1) $\mathcal{S}el$ *is a* $\mathrm{P}((\emptyset \times \mathrm{Fin})(\mathcal{A}))$.
2) *There is* $k \in \omega$ *s. t. there is no* k*-tower of monochromatic functions.*
3) $\mathcal{S}el \subseteq^{\restriction} (\emptyset \times \mathrm{Fin})(\mathcal{A})$.

Critical ideals on $\omega \times \omega$

$\boldsymbol{\mathcal{ED}(\mathcal{A})}$ – supremum of $\{(\mathrm{Fin} \times \emptyset)(\mathcal{A}), \mathcal{S}el(\mathcal{A})\}$

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Theorem

The following statements are equivalent.

1) $\mathcal{ED}(\mathcal{A})$ *is a* $\mathrm{P}(\emptyset \times \mathrm{Fin})$*-ideal.*
2) $\mathcal{ED}(\mathcal{A}) \perp \emptyset \times \mathrm{Fin}$.
3) $\mathcal{ED}(\mathcal{A}) \subseteq^{\restriction} \emptyset \times \mathrm{Fin}$.

Critical ideals on $\omega \times \omega$

$\boldsymbol{\mathcal{ED}(\mathcal{A})}$ – supremum of $\{(\mathrm{Fin} \times \emptyset)(\mathcal{A}), \mathcal{S}el(\mathcal{A})\}$

Theorem

The following statements are equivalent.

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Theorem

The following statements are equivalent.

1) $\mathcal{ED}(\mathcal{A})$ *is a* $\mathrm{P}(\mathrm{Fin} \times \emptyset)$*-ideal.*
2) $\mathcal{S}el(\mathcal{A})$ *is a* $\mathrm{P}(\mathrm{Fin} \times \emptyset)$*-ideal.*
3) $\mathcal{ED}(\mathcal{A}) \perp \mathrm{Fin} \times \emptyset$.
4) $\mathcal{ED}(\mathcal{A}) \subseteq^{\restriction} \mathrm{Fin} \times \emptyset$.
5) $\mathcal{S}el(\mathcal{A}) \subseteq^{\restriction} \mathrm{Fin} \times \emptyset$.

Summary

	P	P($\emptyset \times$ Fin)	P(Fin $\times \emptyset$)	P(Fin $\times$ Fin)	P($\mathcal{ED}$)	P($\mathcal{S}el$)
$\emptyset \times$ Fin	✓	✓	✓	✓	✓	✓
Fin $\times \emptyset$	✗	✗	✓	✓	✓	✗
Fin $\times$ Fin	✗	✗	✓	✓	✓	✗
$\mathcal{S}el$	✗	✓	✗	✓	✓	✓
$\mathcal{ED}$	✗	✗	✗	✓	✓	✗

	P	P($\emptyset \times$ Fin)	P(Fin $\times \emptyset$)	P(Fin $\times$ Fin)	P($\mathcal{ED}$)	P($\mathcal{S}el$)
$(\emptyset \times \text{Fin})(\mathcal{A})$	✓	✓	✓	✓	✓	✓
$(\text{Fin} \times \emptyset)(\mathcal{A})$	✗	$\subseteq^{\restriction}$	$\subseteq^{\restriction}$	$\subseteq^{\restriction}$	$\subseteq^{\restriction}$	$\subseteq^{\restriction}$
$(\text{Fin} \times \text{Fin})(\mathcal{A})$	✗	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
$\mathcal{S}el(\mathcal{A})$	✗	$\subseteq^{\restriction}$	$\subseteq^{\restriction}$	?	?	?
$\mathcal{ED}(\mathcal{A})$	✓	$\perp, \subseteq^{\restriction}$	$\perp, \subseteq^{\restriction}$	?	?	?

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Thank you