Equivalence between models of ZFC, topological and algebraic properties of C^* algebras and their category of modules.

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Overview

1. Motivation and Strategy
2. Basic and model theoretical problems 1.
    - The join irreducibility of the ideal of the compact operators in the Operator Banach Algebra over a separable (infinite dimensional) Hilbert space.
    - The NCF principle.
    - The construction of the forcing which establishes the independence of NCF and ZFC.
    - On their equivalence.
3. Basic and model theoretical problems 2.
    - Ultrapowers of C^* algebras
    - Ultrapowers of C^* algebras and CH
    - Measurable cardinals - existence and non-existence

Overview - Continuation

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Motivation and Strategy: Why?

Weak diamond and SUP are two combinatorial principle, which inspire a lot of people in algebra and analysis - such as the following question?
Let R be a non-perfect (right or left) ring, for instance, a (right/left) P.I.D which is (right/left) Artinian ring then the question if R-projectivity implies projectivity depends on the model of ZFC, i.e. for instance the weak diamond implies that any ring with size at most $2^{\aleph_0}$ and all small modules over it, which are $2^{\aleph_0}$-free are indeed free, see, e.g., the case of being non-perfect a P.I.D .

For arbitrary huge rings and modules over them we can use the statement of the weak diamond on the corresponding regular uncountable cardinal (as a succesor), for instance $2^{\aleph_0} < 2^{\aleph_1}$ is a combinatorial statement equivalent to the weak diamond at $\aleph_1$.

Now, assuming the SUP we can show that, in general, R - proje

Motivation and Strategy: Why?

ctivity does not imply projectivity at all, i.e there exist a module (big one) M ove R which is with $proj.dim(M) = 1$, which is flat and which is R-projective, [**see, for these results, generalizations and applications, and the connections with cotorsion pairs, for instance, the paper of [J. Trlifaj, 1991] , the references given therein and the new reasearch of the author**]

Definition

Let H be a separable infinite dimensional Hilbert space, i.e isometrically isomorphic to l^2. Considering the complete topological ring of all bounded operators over it, we denote this Banach algebra as $B(l^2)$. The two-sided ideal of the compact operators in $B(l^2)$ will be denoted as $\mathcal{K}(l^2)$.

We recall that the ideal $\mathcal{K}(l^2)$ is actually a topological ideal and topologically it is the closure of the set $F(l^2)$ of the finite rank ope-

Irreducibility of these ideals

rators - since l^2 admits a Shauder basis. Unfortunately, although the ideal $\mathcal{K}(l^2)$ is the largest two-sided topologically closed ideal in $B(l^2)$, containing the set $F(l^2)$, there are infinitely many one-sided ideals being only algebraic ideals contained in $\mathcal{K}(l^2)$ and containig $F(l^2)$.

We know that in any unital ring (associative) the lattice of its algebraic ideals (one-sided) two-sided is modular and moreover complete, with join being the "sum" of them and a meet being the intersection. Moreover topologically (the closedness is preserved by the same operations).

Motivation

Definition

1. Given a lattice L we call the element $a \in L$ a join-reducible iff $\exists a_1, a_2 \in L$ such that $a_i \vee a_2 = a$ and $a_i \neq a$, $i = 1, 2$. In this case we say that the join representation $a_i \vee a_2 = a$ is proper.
2. In the terms of this definition if the only representation is the trivial one, i.e at least one of the elements a_i is equal to a then a is called join-irreducible.

Theorem

Under MA the ideal $\mathcal{K}(l^2)$ is join-reducible.

The first proof relies on CH and it is given by [A. Blass and G. Weiss (1978)].

Join irreducibility

Definition

Near Coherence Filters (NCF) is the following combinatorial assertation: Any two free ultrafilters over ω are related (in both directions) by a finite-to-one map from ω to themselfs such that the images over the ultrafilters coincide.

Theorem ([Blass and Shelah (1989)])

Under NCF, $\mathcal{K}(l^2)$ is join-irreducible. Moreover, NCF holds iff $\mathcal{K}(l^2)$ is join-irreducible.

NCF

Lemma

TFCA:

1. *Some shift ideal which contains the shift ideal of the bounded functions is join-irreducible.*
2. *NCF*

Proof.

The proof is actually is reproduction of the proof by using CH, so it is a replique which states that actually we should establish the join-reducibility of the ideals in the first point.
Actually it is enough to use the NCF, which is consistent with and implies $\neg$ CH. Actually the construction is assuming the failure of NCF which implies the join reducibility of all shift ideals containing the bounded functions.

The Miller forcing of superperfect trees

Definition

[**Blass and Shelah (1989)**] Let p be a tree of finite subsets of ω; A node $a \in p$ is said to be infinitely branching in p, it it has infinitely many succesors in the tree. Three p is called superperfect if any has a succesor, which is infinitely branching in p.

Definition

[**Blass and Shelah (1989)**] Let Q be a forcing POSET, we say that Q satisfies Miller condition if $\forall p \in Q$, being the forcing conditions holds that actually they p‘s are superperfect trees, and moreover the withnes of that are the extensions of the conditions (in Q), which actually are superperfect subtrees of the initially given conditions.

The Miller forcing of superperfect trees

that this forcing is proper and that the set of superperfect trees with interval structure is dense in Q as a corollary [Blass and Shelah (1989)] shows that there exists a model - a generating set such that the ground model in that forcing relation implies **NCF.**

Ultrapowers of C^* algebras

In the paper of [**L. Ge and D. Hadwin, 2001**] establishes the folloeing important (model-theoretical) result:

Theorem

*[**L. Ge and D. Hadwin, 2001**] CH is equivalent to the assertion that any to (arbitrary) ultrapowers of a C^* algebras serving as model - structures are isomorphic. Here, ultrapowers stand for a **free ultrafilter** over ω. These ultrapowers are isomorphic, because they all are models of CH and CH+ZFC implies them.*

Ultrapowers of C^* algebras

But the following is of main interest here:

Theorem

[L. Ge and D. Hadwin, 2001] *There exists a measurable cardinal if and only if there exists an ultrapower of separable C^* algebras which is separable, and this will be the required model.*

In particular, the first theorem says, there are no separable ultrapowers and second there is, so by virtue of the second one it follows that there is a model, non isomorphic to some the other models - attention there always exists a non-separable ultrapower so, being separable implies that they cannot be isomorphic even as Banach spaces, so CH cannot be true in that situation. One more proof that measurability contradicts CH (From elementary observations in FA).

Continuation

Theorem

[J.Šaroch.,2015] *Let there no measurable cardinals then in the terms of the previous definitions if $Add(M) \subset Prod(M)$ then M is algebraically compact.*

We know that algebraically compact are always saturated and henceforth they are universal models, so henceforth they are in specific important.

Now,

Measurability

Theorem

TFCA

1. *If the topological product of two topological spaces is extremally disconnected (the projective objects in the category of Hausdorf Compact spaces) then* ***there exists a situation where*** *the two spaces in the product are not discrete.*
2. *Not all dicrete spaces are realcompact top. spaces*
3. *There exist a separable ultrapower of separable* $C^{(}*)$ *algebras.*
4. *There exists a measurable cardinal.*

In particular, the negation of the last theorem implies that set theoretic assumption given by **[J.Šaroch.,2015].**

Thank You!

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