On Δ-spaces X and their characterization in terms of spaces $C_p(X)$

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5. See also R. W. Knight, *Δ-Sets*, Trans. Amer. Math. Soc. 339 (1993), 45-60.

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5. No Δ-set can have cardinality $\mathfrak{c}$, therefore Continuum Hypothesis implies that there is no uncountable Δ-set of reals (Przymusiński).
6. Uncountable Δ-sets of reals exist or not, depending on a model of the set theory.

Definition 1 (Kąkol-Leiderman)

A topological space X is called a Δ-space if for every decreasing sequence $(D_n)_n$ of subsets of X with $\bigcap_n D_n = \emptyset$, there is a decreasing sequence $(V_n)_n$ of open subsets of X, $D_n \subset V_n$ for every $n \in N$ and $\bigcap_n V_n = \emptyset$.

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3. Non-metrizable Δ-spaces exist in ZFC.

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9. Hence $\beta\mathbb{Q}$, $\beta\mathbb{R}$ and $\beta\mathbb{N}$ are not Δ-spaces. What about positive examples ?
10. $C_p(X)$, $C_k(X)$ - spaces of all real-valued cont. functions on Tychonoff X with the pointwise and the compact-open topology, respectively.

1. $C_p(X)$ is called *distinguished* if for each bounded $A \subset \mathbb{R}^X$ there is bounded $B \subset C_p(X)$ with $A \subset \overline{B}$, the closure in $\mathbb{R}^X$ (Ferrando, Kąkol, Leiderman, Saxon).

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Theorem 2

The following statements are equivalent for Tychonoff X:

(a) *Dual $C_p(X)'_\beta = (L_p(X), \beta(L_p(X), C_p(X))$ carries the finest l.c. topology, where $L_p(X) = span\{\delta_x : x \in X\}$.*

(b) *The space $C_p(X)$ is distinguished.*

(c) *$\forall\ f \in \mathbb{R}^X\ \exists$ bounded $B \subset C_p(X)$ with $f \subset \overline{B}$, the closure in $\mathbb{R}^X$ (Ferrando, Kąkol, Leiderman,Saxon).*

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From (c)$\Rightarrow$ Every countable Tychonoff space is a Δ-space.

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Corollary 4

If X is a first-countable compact space, then X is a Δ-space iff X is countable.

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Theorem 5 (Ferrando, Kąkol, Leiderman, Saxon)

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3. For each $n \in \mathbb{N}$, the set $\{n\}$ is open, and for each $A \in \mathfrak{A}$, a base of neighbourhoods of A is the family of all sets $\{A\} \cup (A \setminus F)$, where F runs over all finite subsets of $\mathbb{N}$.

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4. Then $\Psi(\mathfrak{A})$ is a first-countable separable locally compact Tychonoff space.

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3. X is not Eberlein, since every separable Eberlein compact space is metrizable, but $\Psi(\mathfrak{A})$ is metrizable iff $\mathfrak{A}$ is countable.

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Corollary 9

If X is an infinite compact space and $X \in \Delta$, then the Banach space $C(X)$ is not a Grothendieck space. The converse fails, as $X = [0, \omega_1]$ applies.

Each scattered metrizable X is a Δ-space: Every such X is Čech-complete (Tkachuk), and apply Theorem 5. There exist scattered X not being Čech-complete; every second countable scattered is Čech-complete (Knaster, Urbanik).

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Corollary 11

(1) *The existence of an uncountable sep. metrizable Δ-space is independent of ZFC.*

(2) *There is an uncountable sep. metrizable Δ-space iff there is a sep. countably paracompact non-normal Moore space.*

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Cont. images of Lindelöf Čech-complete Δ-spaces are Δ-sp.

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3. In Theorem 15 linearity of T cannot be dropped .

1. Indeed, Y - metrizable separable, $|Y| = \mathfrak{c}$ s.t. $C_k(Y)$ is analytic, i.e. $C_k(Y)$ is a continuous image of irrationals J. Let $S \simeq \{0\} \cup \{n^{-1} : n \in \mathbb{N}\}$. $S \in \Delta$, $Y \notin \Delta$!

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2. $C_p(S)$ contains a closed homeom. copy of J. There is a continuous surjection $L : J \to C_k(Y)$ which extends to a continuous (not linear) surjection $T : C_p(S) \to C_k(Y)$.

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Theorem 16 (Kąkol, Michalak, Leiderman)

If X and Y are inf. Tychonoff and there is a seq. continuous linear surjection $T : C_p(X) \to C_k(Y)_w$, every compact set in Y is finite, where $C_k(Y)_w$ endowed with the weak topology.

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Assume that Y is ℓ-dominated by X. If X is Eberlein compact, Y is Eberlein compact. If X is scattered Eberlein compact, Y is scattered Eberlein compact.

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Let X and Y be metrizable and Y is ℓ-dominated by X. If X is scattered, then Y is scattered.

3. It is unknown if metrizability of X and Y can be dropped.

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Find scattered compact X, Y s.t. $X \in \Delta$, $Y \notin \Delta$ and there exists a continuous linear surjection from $C(X)$ onto $C(Y)$.

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Problem 21

Characterize compact Δ-spaces X in terms of suitable topological properties of the Banach space $C(X)$ or its dual.

3. If X is a compact Δ-space, then $C(X)$ is Asplund.

Theorem 22

Assume that Y is ℓ-dominated by X. If X is a Δ-space, then Y also is a Δ-space.

Lemma 23

Let X and Y be two sets and let $E \subset \mathbb{R}^X$ and $F \subset \mathbb{R}^Y$ be dense vector subspaces of $\mathbb{R}^X$ and $\mathbb{R}^Y$, respectively. Assume that $T : E \longrightarrow F$ is a continuous linear surjection between lcs E and F. Then T admits a continuous linear surjective (unique) extension $\widehat{T} : \mathbb{R}^X \longrightarrow \mathbb{R}^Y$.

Proof. We need some facts:

Property 1. Every closed vector subspace H of $\mathbb{R}^X$ is complemented in $\mathbb{R}^X$ and the quotient $\mathbb{R}^X/H$ is linearly homeomorphic to the product $\mathbb{R}^Z$ for some set Z .

Property 2. The product topology on $\mathbb{R}^X$ is minimal, i.e. $\mathbb{R}^X$ does not admit a weaker Hausdorff locally convex topology.

Property 3. $\mathbb{R}^Y$ fulfills the extension property, i.e. if M is a vector subspace of a lcs L, then every continuous linear mapping $T : M \longrightarrow \mathbb{R}^Y$ admits a continuous linear extension $\widehat{T} : L \longrightarrow \mathbb{R}^Y$. By Property 3, there exists a continuous linear extension $\widehat{T} : \mathbb{R}^X \longrightarrow \mathbb{R}^Y$ of T such that $F \subset \widehat{T}(\mathbb{R}^X)$. We prove that $\widehat{T}$ is a surjective mapping.

Denote by $\varphi : \mathbb{R}^X / \ker(\widehat{T}) \longrightarrow \mathbb{R}^Y$ the injective mapping associated with the quotient mapping $Q : \mathbb{R}^X \longrightarrow \mathbb{R}^X / \ker(\widehat{T})$, where $\ker(\widehat{T})$ is the kernel of $\widehat{T}$ and $\varphi \circ Q = \widehat{T}$. By Property 1, the space $\mathbb{R}^X / \ker(\widehat{T})$ is linearly homeomorphic to the product $\mathbb{R}^Z$ for some set Z. So we may assume that φ is a continuous linear bijection from $\mathbb{R}^Z$ onto a dense subspace $\widehat{T}(\mathbb{R}^X)$ of $\mathbb{R}^Y$. This implies that on $\widehat{T}(\mathbb{R}^X)$ there exists a stronger locally convex topology ξ such that $(\widehat{T}(\mathbb{R}^X), \xi)$ is linearly homeomorphic with $\mathbb{R}^Z$. However, by Property 2, $\mathbb{R}^Z$ does not admit a weaker Hausdorff locally convex topology, hence $\widehat{T}(\mathbb{R}^X)$ is isomorphic to the complete lcs $\mathbb{R}^Z$. Finally, $\widehat{T}(\mathbb{R}^X)$ is closed in $\mathbb{R}^Y$ and then $\widehat{T}$ is a surjection.

First Proof. Let $T : C_p(X) \longrightarrow C_p(Y)$ be a continuous linear surjection. Denote by $\widehat{T} : \mathbb{R}^X \longrightarrow \mathbb{R}^Y$ the extension of T which is supplied by Lemma 23. By Theorem 2, $C_p(X)$ is distinguished and we can apply item (c) of Theorem 2. Take arbitrary $f \in \mathbb{R}^Y$. There exists $g \in \mathbb{R}^X$ with $\widehat{T}(g) = f$. Then there exists a bounded set $B \subset C_p(X)$ such that $g \in \overline{B}^{\mathbb{R}^X}$. We define $A = T(B)$. It is easy to see that A is bounded and $f \in \overline{A}^{\mathbb{R}^Y}$ which means that $C_p(Y)$ is distinguished, equivalently, Y is a Δ-space, by Theorem 2.

Second Proof. If $T : C_p(X) \longrightarrow C_p(Y)$ is a continuous linear surjection, then the adjoint mapping $T^* : (L_p(Y), \beta_Y) \longrightarrow (L_p(X), \beta_X)$ is continuous and injective, where β_X and β_Y are the strong topologies on the duals $L_p(X)$ and $L_p(Y)$, respectively. Denote by $Z = T^*(L_p(Y))$. Endow Z with the induced topology $\beta_X\restriction_Z$. Since $T^* : (L_p(Y), \beta_Y) \to (Z, \beta_X|Z)$ is a continuous linear bijection, the sets $T^*(U)$, where U run over all absolutely convex neighbourhoods of zero in $(L_p(Y), \beta_Y)$, form a base of absolutely convex neighbourhoods of zero for a locally convex topology ξ on X such that $\beta_X\restriction_Z \leq \xi$ and $T^* : (L_p(Y), \beta_Y) \longrightarrow (Z, \xi)$ is a linear homeomorphism.

Since $C_p(X)$ is distinguished, the topology β_X is the finest locally convex topology, by item (c) of Theorem 2. The property of having the finest locally convex topology is inherited by vector subspaces, so the induced topology $\beta_X \upharpoonright_Z$ is the finest locally convex one. Then $\beta_X \upharpoonright_Z = \xi$ is the finest locally convex topology, so β_Y is of the same type on $L_p(Y)$. Hence $C_p(Y)$ is distinguished, by Theorem 2, equivalently, Y is a Δ-space, again by Theorem 2.

Several open problems have been posed in the following direction: Suppose that a dense subspace of $C_p(X)$ is a "nice" (not necessarily linear) continuous image of $\mathbb{R}^\kappa$, for some cardinal κ; must X be discrete? Lemma 23 implies immediately

Corollary 24

Let a dense subspace of $C_p(X)$ be a continuous linear image of $\mathbb{R}^\kappa$, for some cardinal κ. Then X is discrete.

For simplicity, a topological space X is called a *Q-space* if each subset of X is F_σ, or, equivalently, each subset of X is G_δ in X.

Theorem 25

X, Y - normal. Assume Y is l-dominated by X. If X is a Q-space, then Y is a Q-space.

Proof. Normal X is a Q-space iff X is *strongly splittable*, i.e. for every $f \in \mathbb{R}^X$ there exists a sequence $S = \{f_n : n \in \omega\} \subset C_p(X)$ such that $f_n \to f$ in $\mathbb{R}^X$. Let $T : C_p(X) \longrightarrow C_p(Y)$ be a continuous linear surjection. Denote by $\widehat{T} : \mathbb{R}^X \longrightarrow \mathbb{R}^Y$ the extension of T which is supplied by Lemma 23. Take arbitrary $f \in \mathbb{R}^Y$. There exists $g \in \mathbb{R}^X$ with $\widehat{T}(g) = f$. Then there exists a sequence $B \subset C_p(X)$ converging to g in $\mathbb{R}^X$. We define $A = T(B)$. It is easy to see that $A \subset C_p(Y)$ converges to f in $\mathbb{R}^Y$.