WINTER SCHOOL 2022 PROBLEM SESSION

During the 2022 Winter School in Abstract Analysis (Set Theory and Topology Section), there was a short problem session in which participants were invited to share questions or open problems. Four problems were presented. Here is a brief summary of them (and a solution to one of them that was found during the Winter School).

Question 1. (Noé de Rancourt) Let $S : 2^\omega \to 2^\omega$ be the shift function, defined by

$$S(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$$

Consider sequences $\vec{A} = \langle A_n \mid n < \omega \rangle$ such that, for all $n < \omega$:

(1) $A_{n+1} \subseteq A_n \subseteq 2^\omega$;
(2) S^n induces a bijection from A_n to 2^ω;
(3) $\bigcap_{n<\omega} A_n = \emptyset$.

One can build a sequence satisfying (1)–(3) using the axiom of choice.

Problem: Are there *definable* sequences satisfying (1)–(3)?

Prizes:

- Construction of a Borel example: 5 beers
- Construction of an analytic or coanalytic example: 4 beers
- Construction of a Baire or Lebesgue measurable example: 3 beers
- Proof that there is no Borel example: 1 box of Chapon chocolates

Solution (by Marcin Sabok): Let A_n be the set of all $x \in 2^\omega$ satisfying one of the two following conditions:

- for all $i < n$, $x_i = 1$, and $\{j \in \omega \mid x_j = 1\}$ is finite;
- for all $i < n$, $x_i = 0$, and $\{j \in \omega \mid x_j = 1\}$ is infinite.

It is easy to see that the sequence $\langle A_n \mid n < \omega \rangle$ is a Borel example.

Question 2. (Thilo Weinert) Let ld denote the binary logarithm. Define a function H on the interval $(0, 1)$ by letting

$$H(x) = -\mathrm{ld}(x)x - \mathrm{ld}(1 - x)(1 - x).$$

Problem: Is it the case that $H(y) \leq x$ for all $x \in (0, 1)$, where

$$y = \frac{(2 - \sqrt{2})x}{2\mathrm{ld}\left(1 + \frac{1}{x}\right)}?$$

Notes: Using a computer to graph the function yields strong visual evidence that the answer is positive. This problem is related to the talk that Thilo delivered at this Winter School. See his slides for more context.

Question 3. (Marcin Sabok) Given a graph $G = (V, E)$, a *proper edge-coloring* of G is a function c with domain E such that, for all distinct edges $u, v \in E$, if $u \cap v \neq \emptyset$, then $c(u) \neq c(v)$. The *edge chromatic number* of G, denoted $\chi'(G)$, is the least cardinal χ for which there exists a proper edge coloring $c : E \to \chi$. If G is a Borel graph, then we can similarly define the *measurable edge chromatic number* of G, denoted $\chi'_M(G)$, as the least cardinal χ for which there exists a measurable proper edge coloring $c : E \to \chi$.

Problem: Let G be a hyperfinite bipartite one-ended regular graphing with degree d. Is it true that $\chi'_m(G) = d$?

Notes: For definitions of the terminology in the question, see the slides from Marcin's tutorial series at this Winter School.

Question 4. (Chris Lambie-Hanson) Given a set X, we say that a coloring $c : [X]^2 \to \omega$ is *triangle-free* if, for all distinct $x, y, z \in X$, it is not the case that $c(x, y) = c(x, z) = c(y, z)$. If $c : [X]^2 \to \omega$ is triangle-free, we say that it is a *maximal triangle-free coloring* if, for every $Y \supsetneq X$ and every $d : [Y]^2 \to \omega$ extending c, d is *not* triangle-free.

Problem: What is the minimal cardinality of a set X such that there exists a maximal triangle-free coloring $c : [X]^2 \to \omega$? In particular, is it consistent that this cardinality is strictly less than 2^ω?

Notes: Here are some preliminary observations.

- If $c : [X]^2 \to \omega$ is a maximal triangle-free coloring, then X is uncountable.
- By the Erdős-Rado theorem, if $|X| > 2^\omega$ and $c : [X]^2 \to \omega$, then c is not triangle-free.
- The coloring $c : [{}^\omega 2]^2 \to \omega$ defined by $c(f, g) = \min\{n < \omega \mid f(n) \neq g(n)\}$ is a maximal triangle-free coloring.
- If, in the problem, "triangle-free" is replaced by "odd-cycle-free", then the answer is known to be 2^ω.