Nontrivial coherent families of functions

Lecture 3

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III. Higher dimensions

Review

Let $n \geq 2$. For $\vec{f} \in ({}^\omega\omega)^n$, $I(\vec{f}) = I(f_0) \cap \ldots \cap I(f_{n-1})$.
Let $\mathcal{F} \subseteq {}^\omega\omega$, and let $\Phi = \langle \varphi_{\vec{f}} : I(\vec{f}) \to \mathbb{Z} \mid \vec{f} \in \mathcal{F}^n \rangle$ be an alternating family of functions.

Φ is *n-coherent* if, for all $\vec{f} \in \mathcal{F}^{n+1}$, we have $\sum_{i=0}^{n} (-1)^i \varphi_{\vec{f}^i} =^* 0$.
For $n = 2$, this becomes $\varphi_{gh} - \varphi_{fh} + \varphi_{fg} =^* 0$ for all $f, g, h \in \mathcal{F}$.

Φ is *n-trivial* if there is an alternating family

$$\left\langle \psi_{\vec{f}} : I(\wedge \vec{f}) \to \mathbb{Z} \,\middle|\, \vec{f} \in \mathcal{F}^{n-1} \right\rangle$$

such that $\sum_{i=0}^{n-1} (-1)^i \psi_{\vec{f}^i} =^* \varphi_{\vec{f}}$ for all $\vec{f} \in ({}^\omega\omega)^n$.
For $n = 2$, this is a family $\langle \psi_f \mid f \in \mathcal{F} \rangle$ such that $\varphi_{fg} =^* \psi_g - \psi_f$ for all $f, g \in \mathcal{F}$.
$\lim^n \mathbf{A} = 0$ iff every n-coherent family on $({}^\omega\omega)^n$ is n-trivial.

Two basic lemmata

Lemma

Suppose that $\mathcal{F} \subseteq {}^\omega\omega$ *and* $\Phi = \langle \varphi_f \mid f \in \mathcal{F} \rangle$ *is coherent. Then the following are equivalent:*

1. Φ *is trivial;*
2. *there is a family* $\langle \psi_f : I(f) \to \mathbb{Z} \mid f \in \mathcal{F} \rangle$ *of finitely supported functions such that, for all* $f, g \in \mathcal{F}$,

$$\varphi_f - \psi_f = \varphi_g - \psi_g.$$

Two basic lemmata

Lemma

Suppose that $\mathcal{F} \subseteq {}^\omega\omega$, $n > 1$, *and* $\Phi = \langle \varphi_{\vec{f}} \mid \vec{f} \in \mathcal{F}^n \rangle$ *is n-coherent. Then the following are equivalent:*

1. Φ *is trivial;*
2. *there is an alternating family* $\langle \tau_{\vec{f}} : I(\vec{f}) \to \mathbb{Z} \mid \vec{f} \in \mathcal{F}^n \rangle$ *of finitely supported functions such that, for all* $\vec{f} \in \mathcal{F}^{n+1}$, $\sum_{i=0}^{n} (-1)^i (\varphi_{\vec{f}^i} - \tau_{\vec{f}^i}) = 0$.

Sketch of proof ($n = 2$).

$1 \Rightarrow 2$: If $\langle \psi_f \mid f \in \mathcal{F} \rangle$ witnesses that Φ is trivial, then, for all $f, g \in \mathcal{F}$, let $\tau_{fg} = \varphi_{fg} - (\psi_g - \psi_f)$.

$2 \Rightarrow 1$: Given $\langle \tau_{fg} \mid f, g \in \mathcal{F} \rangle$: for each $x \in \omega^2$, find $f_x \in \mathcal{F}$ such that $x \in I(f_x)$. For all $f \in \mathcal{F}$ and all $x \in I(f)$, let $\psi_f(x) = (\tau_{f,f_x}(x) - \varphi_{f,f_x}(x))$. □

Lemma

Suppose that $\mathcal{F} \subseteq {}^{\omega}\omega$ *is countable and* $\Phi = \langle \varphi_f \mid f \in \mathcal{F} \rangle$ *is coherent. Then* Φ *is trivial.*

Lemma

Suppose that $n \geq 1$, $\mathcal{F} \subseteq {}^{\omega}\omega$, $|\mathcal{F}| < \aleph_n$, and $\Phi = \langle \varphi_{\vec{f}} \mid \vec{f} \in \mathcal{F}^n \rangle$ is n-coherent. Then Φ is trivial.

Proof for $n = 2$.

Enumerate $\mathcal{F}$ as $\langle f_\alpha \mid \alpha < \omega_1 \rangle$ and, for each $\alpha < \beta < \omega_1$, denote $\varphi_{f_\alpha f_\beta}$ by $\varphi_{\alpha\beta}$. By recursion on $\alpha < \omega_1$, we will define functions $\psi_\alpha : I(f_\alpha) \to \mathbb{Z}$ such that, for all $\alpha < \beta < \omega_1$, we have $\psi_\beta - \psi_\alpha =^* \varphi_{\alpha\beta}$. Suppose that $\beta < \omega_1$ and we have defined $\langle \psi_\alpha \mid \alpha < \beta \rangle$. For each $\alpha < \beta$, let $\tau_\alpha = \varphi_{\alpha\beta} + \psi_\alpha$.

Claim: $\langle \tau_\alpha \mid \alpha < \beta \rangle$ is 1-coherent.
Proof of claim: For all $\alpha < \alpha' < \beta$, we have

$$\begin{aligned}
\tau_{\alpha'} - \tau_\alpha &= \varphi_{\alpha'\beta} + \psi_{\alpha'} - \varphi_{\alpha\beta} - \psi_\alpha \\
&= \varphi_{\alpha'\beta} - \varphi_{\alpha\beta} + (\psi_{\alpha'} - \psi_\alpha) \\
&=^* \varphi_{\alpha'\beta} - \varphi_{\alpha\beta} + \varphi_{\alpha\alpha'} \\
&=^* 0.
\end{aligned}$$

Proof (cont.)

Since β is countable, $\langle \tau_\alpha \mid \alpha < \beta \rangle$ is trivial, so we can let $\psi_\beta : I(f_\beta) \to \mathbb{Z}$ trivialize it.

Then, for all $\alpha < \beta$, we have

$$\begin{aligned} \psi_\beta - \psi_\alpha &=^* \tau_\alpha - \psi_\alpha \\ &= (\varphi_{\alpha\beta} + \psi_\alpha) - \psi_\alpha \\ &= \varphi_{\alpha\beta}. \end{aligned}$$

Thus, ψ_β is as desired, and we can continue with our construction. At the end, we have arranged that $\langle \psi_\alpha \mid \alpha < \omega_1 \rangle$ trivializes Φ. □

Corollary

For all $n > 1$, if $\mathfrak{d} < \aleph_n$, then $\lim^n \mathbf{A} = 0$.

Theorem (Dow-Simon-Vaughan, '89)

If $\mathfrak{d} = \aleph_1$, *then* $\lim^1 \mathbf{A} \neq 0$.

Theorem (Bergfalk, '17, [2])

Suppose that $\mathfrak{b} = \mathfrak{d} = \aleph_2$ *and* $\diamondsuit(S^{\aleph_2}_{\aleph_1})$ *holds. Then* $\lim^2 \mathbf{A} \neq 0$.

Proof sketch.

Fix a sequence $\langle f_\alpha \mid \alpha < \omega_2 \rangle$ that is $<^*$-increasing and $<^*$-cofinal in ${}^\omega\omega$. It will suffice to construct a nontrivial 2-coherent family $\langle \varphi_{\alpha\beta} : I(f_\alpha \wedge f_\beta) \to \mathbb{Z} \mid \alpha < \beta < \omega_2 \rangle$.

By $\diamondsuit(S^{\aleph_2}_{\aleph_1})$, we can fix a sequence of sequences

$$\langle \langle \psi^\beta_\alpha : I(f_\alpha) \to \mathbb{Z} \mid \alpha < \beta \rangle \mid \beta \in S^{\aleph_2}_{\aleph_1} \rangle$$

such that, for every sequence $\langle \psi_\alpha : I(f_\alpha) \to \mathbb{Z} \mid \alpha < \omega_2 \rangle$, there are stationarily many $\beta \in S^{\aleph_2}_{\aleph_1}$ such that

$$\langle \psi_\alpha \mid \alpha < \beta \rangle = \langle \psi^\beta_\alpha \mid \alpha < \beta \rangle.$$

Proof (cont.)

We now construct $\langle \varphi_{\alpha\beta} \mid \alpha < \beta < \omega_2 \rangle$ by recursion on β. Suppose that $\beta < \omega_2$ and $\langle \varphi_{\alpha\alpha'} \mid \alpha < \alpha' < \beta \rangle$ has been defined.

Case 1: $\beta \in S^{\aleph_2}_{\aleph_1}$ and $\langle \psi^\beta_\alpha \mid \alpha < \beta \rangle$ 2-trivializes $\langle \varphi_{\alpha\alpha'} \mid \alpha < \alpha' < \beta \rangle$. Since $\mathrm{cf}(\beta) = \omega_1$, by a construction from the first lecture, we can find a nontrivial 1-coherent family of functions $\langle \tau^\beta_\alpha : I(f_\alpha) \to \mathbb{Z} \mid \alpha < \beta \rangle$. Now let $\varphi_{\alpha\beta} = -\psi^\beta_\alpha - \tau^\beta_\alpha$ for all $\alpha < \beta$.

Claim: This maintains 2-coherence.
Proof of claim: For all $\alpha < \alpha' < \beta$, we have

$$\begin{aligned}
\varphi_{\alpha'\beta} - \varphi_{\alpha\beta} + \varphi_{\alpha\alpha'} &= -\psi^\beta_{\alpha'} - \tau^\beta_{\alpha'} + \psi^\beta_\alpha + \tau^\beta_\alpha + \varphi_{\alpha\alpha'} \\
&= -(\psi^\beta_{\alpha'} - \psi^\beta_\alpha) - (\tau^\beta_{\alpha'} - \tau^\beta_\alpha) + \varphi_{\alpha\alpha'} \\
&=^* -\varphi_{\alpha\alpha'} - 0 + \varphi_{\alpha\alpha'} \\
&= 0.
\end{aligned}$$

Proof (conclusion).

Case 2: Otherwise. Since $\langle \varphi_{\alpha\alpha'} \mid \alpha < \alpha' < \beta \rangle$ is 2-coherent and $\beta < \omega_2$, it is also 2-trivial. Let $\langle -\varphi_{\alpha\beta} \mid \alpha < \beta \rangle$ be an arbitrary witness to its triviality.

Claim: $\langle \varphi_{\alpha\beta} \mid \alpha < \beta < \omega_2 \rangle$ is nontrivial.

Suppose for sake of contradiction that $\langle \psi_\alpha \mid \alpha < \omega_2 \rangle$ satisfies $\psi_\beta - \psi_\alpha =^* \varphi_{\alpha\beta}$ for all $\alpha < \beta < \omega_2$. Find $\beta \in S^{\aleph_2}_{\aleph_1}$ such that $\langle \psi_\alpha \mid \alpha < \beta \rangle = \langle \psi^\beta_\alpha \mid \alpha < \beta \rangle$. Then at β we were in Case 1 of the construction. Therefore, for all $\alpha < \beta$, we have

$$\psi_\beta - \psi_\alpha =^* \varphi_{\alpha\beta} = -\psi_\alpha - \tau^\beta_\alpha$$

so $-\psi_\beta =^* \tau^\beta_\alpha$, i.e., $-\psi_\beta$ trivializes $\langle \tau^\beta_\alpha \mid \alpha < \beta \rangle$, contradicting the fact that $\langle \tau^\beta_\alpha \mid \alpha < \beta \rangle$ is nontrivial. $\square$

Consistent nonvanishing

Corollary (Bergfalk, '17, [2])

$\mathrm{PFA} \Rightarrow \lim^2 \mathbf{A} \neq 0$.

Recall that $\mathrm{PFA} \Rightarrow \lim^n \mathbf{A} = 0$ for all $n \in \omega \setminus \{0, 2\}$.

Theorem (Veličković-Vignati, '21, [5])

Let $n \geq 1$. If $\mathfrak{b} = \mathfrak{d} = \aleph_n$ and $\mathrm{w}\diamondsuit(S^{\aleph_{k+1}}_{\aleph_k})$ holds for all $k < n$, then $\lim^n \mathbf{A} \neq 0$.

Therefore, for any $n \geq 1$, it is consistent with ZFC that $\lim^n \mathbf{A} \neq 0$.

Simultaneous nonvanishing

Theorem (Bergfalk-LH, '21, [4])

(Assuming the consistency of a weakly compact cardinal), it is consistent that $\lim^n \mathbf{A} = 0$ *for all* $n \geq 1$ *(simultaneously).*

More precisely, if κ *is a weakly compact cardinal and* $\mathbb{P}$ *is a length-*κ *finite support iteration of Hechler forcing, then, in* $V^{\mathbb{P}}$, $\lim^n \mathbf{A} = 0$ *for all* $n \geq 1$.

Sketch of proof (n=2).

$\mathbb{P}$ adds a sequence $\langle f_\alpha \mid \alpha < \kappa \rangle$ that is $<^*$-increasing and $<^*$-cofinal in ${}^\omega\omega$. Fix $p \in \mathbb{P}$ and a name $\dot{\Phi} = \langle \dot{\varphi}_{\dot{f}\dot{g}} \mid \dot{f}, \dot{g} \in {}^\omega\omega \rangle$ for a 2-coherent family. We will find $q \leq p$ that forces $\dot{\Phi}$ to be 2-trivial.

It suffices to show that q forces the existence of an unbounded $\dot{A} \subseteq \kappa$ such that $\langle \dot{\varphi}_{\alpha\beta} \mid \alpha < \beta \in \dot{A} \rangle$ is trivial, where $\dot{\varphi}_{\alpha\beta}$ denotes $\dot{\varphi}_{f_\alpha f_\beta}$.

In turn, it suffices to show that q forces the existence of a family $\langle \dot{\psi}_{\alpha\beta} \mid \alpha < \beta \in \dot{A} \rangle$ of finitely supported functions such that

$$(\dot{\varphi}_{\beta\gamma} - \dot{\psi}_{\beta\gamma}) - (\dot{\varphi}_{\alpha\gamma} - \dot{\psi}_{\alpha\gamma}) + (\dot{\varphi}_{\alpha\beta} - \dot{\psi}_{\alpha\beta}) = 0$$

for all $\alpha < \beta < \gamma \in \dot{A}$. Let $\dot{e}(\alpha, \beta, \gamma)$ denote $\dot{\varphi}_{\beta\gamma} - \dot{\varphi}_{\alpha\gamma} + \dot{\varphi}_{\alpha\beta}$. The above equation then becomes

$$\dot{e}(\alpha, \beta, \gamma) = \dot{\psi}_{\beta\gamma} - \dot{\psi}_{\alpha\gamma} + \dot{\psi}_{\alpha\beta}.$$

Proof sketch (cont.)

For all $\alpha < \beta < \gamma < \kappa$, $\dot{e}(\alpha, \beta, \gamma)$ is forced to be a finitely supported function, so we can find $q_{\alpha\beta\gamma} \leq p$ deciding the value of the restriction of $\dot{e}(\alpha, \beta, \gamma)$ to its support, say as $\mathrm{e}(\alpha, \beta, \gamma)$. We can also arrange that $q_{\alpha\beta\gamma} \Vdash$ "$\dot{f}_\alpha \leq \dot{f}_\beta \leq \dot{f}_\gamma$".

Using the weak compactness of κ, we can find an unbounded $H \subseteq \kappa$ and a finite partial function $\mathrm{e}^* : \omega^2 \to \mathbb{Z}$ such that

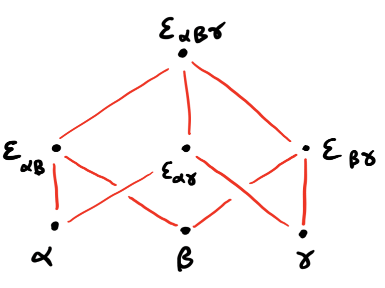
- for all $\alpha < \beta < \gamma \in H$, we have $\mathrm{e}(\alpha, \beta, \gamma) = \mathrm{e}^*$;
- the sequence $\langle q_{\alpha\beta\gamma} \mid \alpha < \beta < \gamma \in H \rangle$ exhibits some strong uniformities (in particular, it forms a kind of 3-dimensional Δ-system, with a "root", $q_\emptyset$).

Proof (cont.)

$q_\emptyset$ then forces the existence of

- an unbounded $\dot{A} \subseteq H$;
- for each $\alpha < \beta \in \dot{A}$, an ordinal $\dot{\varepsilon}_{\alpha\beta} \in H \setminus (\beta + 1)$;
- for each $\alpha < \beta < \gamma \in \dot{A}$, an ordinal $\dot{\varepsilon}_{\alpha\beta\gamma} \in H \setminus (\max\{\varepsilon_{\alpha\beta}, \varepsilon_{\alpha\gamma}, \varepsilon_{\beta\gamma}) + 1)$

such that, for all $\alpha < \beta < \gamma \in \dot{A}$, the conditions $q_{\alpha,\varepsilon_{\alpha\beta},\varepsilon_{\alpha\beta\gamma}}$, $q_{\alpha,\varepsilon_{\alpha\gamma},\varepsilon_{\alpha\beta\gamma}}$, $q_{\beta,\varepsilon_{\alpha\beta},\varepsilon_{\alpha\beta\gamma}}$, $q_{\beta,\varepsilon_{\beta\gamma},\varepsilon_{\alpha\beta\gamma}}$, $q_{\gamma,\varepsilon_{\alpha\gamma},\varepsilon_{\alpha\beta\gamma}}$, and $q_{\gamma,\varepsilon_{\beta\gamma},\varepsilon_{\alpha\beta\gamma}}$ are all in $\dot{G}$ (the generic filter).



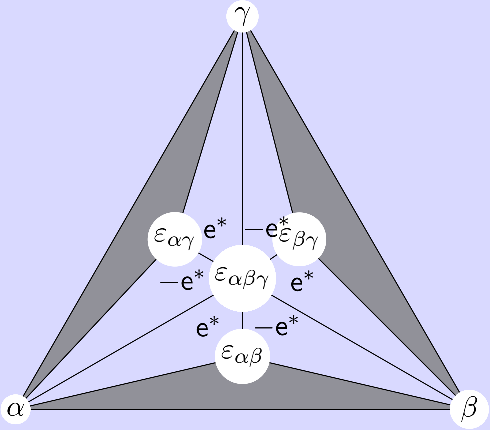
Proof (cont.)

Let G be $\mathbb{P}$-generic over V with $q_\emptyset \in G$. Let A, $\langle \varepsilon_{\alpha\beta} \mid \alpha < \beta \in A\rangle$, and $\langle \varepsilon_{\alpha\beta\gamma} \mid \alpha < \beta < \gamma \in A\rangle$ be as on the previous slide. It suffices

to show that $\langle \varphi_{\alpha\beta} \mid \alpha < \beta \in A\rangle$ is trivial. To this end, for all $\alpha < \beta \in A$, let $\psi_{\alpha\beta} = e(\alpha, \beta, \varepsilon_{\alpha\beta}) = \varphi_{\beta\varepsilon_{\alpha\beta}} - \varphi_{\alpha\varepsilon_{\alpha\beta}} + \varphi_{\alpha\beta}$.

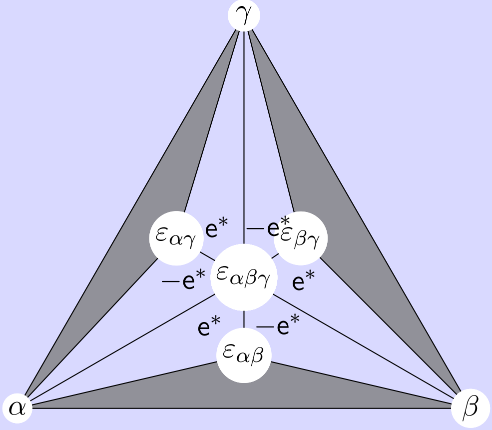
Each $\psi_{\alpha\beta}$ is finitely supported, by the 2-coherence of Φ. We claim that it witnesses that $\langle \varphi_{\alpha\beta} \mid \alpha < \beta \in A\rangle$ is trivial.
We must show that, for all $\alpha < \beta < \gamma \in A$, we have

$$e(\alpha, \beta, \gamma) = \psi_{\beta\gamma} - \psi_{\alpha\gamma} + \psi_{\alpha\beta}.$$



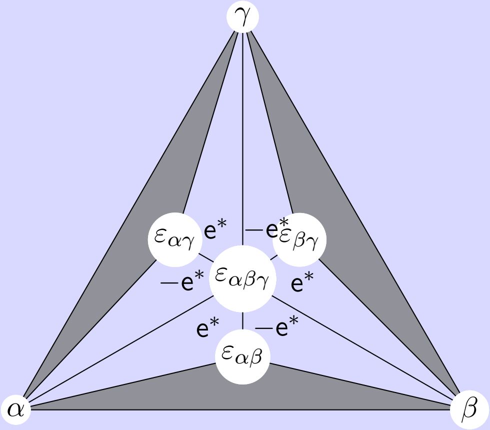
Proof (cont.)

Fix $\alpha < \beta < \gamma \in A$. Adding up all of the $e(\cdots)$-values corresponding to the interior triangles in the above figure (oriented counterclockwise) yields, through cancellation of like terms, precisely $e(\alpha, \beta, \gamma)$.



Proof (cont.)

The $e(\cdots)$-values of the blue interior triangles all equal $\pm e^*$, as their corresponding $q_{\dots}$ conditions are all in G. These values cancel, leaving only $e(\beta, \gamma, \varepsilon_{\beta\gamma}) + e(\gamma, \alpha, \varepsilon_{\alpha\gamma}) + e(\alpha, \beta, \varepsilon_{\alpha\beta})$



Proof (cont.)

$$\begin{aligned} e(\alpha, \beta, \gamma) &= e(\beta, \gamma, \varepsilon_{\beta\gamma}) + e(\gamma, \alpha, \varepsilon_{\alpha\gamma}) + e(\alpha, \beta, \varepsilon_{\alpha\beta}) \\ &= e(\beta, \gamma, \varepsilon_{\beta\gamma}) - e(\alpha, \gamma, \varepsilon_{\alpha\gamma}) + e(\alpha, \beta, \varepsilon_{\alpha\beta}) \\ &= \psi_{\beta\gamma} - \psi_{\alpha\gamma} + \psi_{\alpha\beta}. \end{aligned}$$

□

Further results

Theorem (Bannister-Bergfalk-Moore, '2X, [1])

In the model obtained by adding weakly compact-many Hechler reals, strong homology is additive on the class of locally compact separable metric spaces.

Theorem (Bergfalk-Hrušák-LH, '2X, [3])

Let $\mathbb{P}$ be the forcing to add $\beth_\omega$-many Cohen reals. Then, in $V^{\mathbb{P}}$, $\lim^n \mathbf{A} = 0$ for all $n \geq 1$.

More general systems

Suppose that κ and λ are infinite cardinals. Given a function $f : \kappa \to [\lambda]^{<\omega}$, let $I(f)$ denote the set $\{(i,j) \in \kappa \times \lambda \mid j \in f(i)\}$. We can then define an inverse system $\mathbf{A}_{\kappa,\lambda} = \langle A_f, \pi_{fg} \mid f, g : \kappa \to [\lambda]^{<\omega},\ f \leq g \rangle$, where

- $f \leq g$ if, for all $i \in \kappa$, $f(i) \subseteq g(i)$;
- for all $f : \kappa \to [\lambda]^{<\omega}$, $A_f = \bigoplus_{I(f)} \mathbb{Z}$;
- the maps π_{fg} are the obvious projection maps.

  Note that our original system $\mathbf{A}$ is isomorphic to a cofinal subsystem of $\mathbf{A}_{\aleph_0,\aleph_0}$.

More general systems

Theorem (Bergfalk-LH, '2X)

If κ and λ are infinite cardinals, with $\lambda > \aleph_0$, then $\lim^1 \mathbf{A}_{\kappa,\lambda} \neq 0$.

Corollary (Bergfalk-LH, '2X)

Let $X^n_{\omega_1}$ denote the generalized n-dimensional infinite earring space, i.e., the one-point compactification of the sum of ω_1-many copies of the n-dimensional open unit ball. Then $X^2_{\omega_1}$ (together with countable disjoint unions thereof) is a ZFC counterexample to the additivity of strong homology.

Corollary (Bergfalk-LH, '2X)

The category of pro-abelian groups does not embed fully faithfully into the category of condensed abelian groups (in the context of derived categories).

Questions

Question

Is $\aleph_{\omega+1}$ the minimum value of the continuum consistent with the statement '$\lim^n \mathbf{A} = 0$ for all $n \geq 1$'?

Question

For $n \geq 2$, if $\mathfrak{b} = \mathfrak{d} = \aleph_n$, must it be the case that $\lim^n \mathbf{A} \neq 0$?

Question

What can be said about $\lim^n \mathbf{A}_{\kappa,\lambda}$ for $n \geq 2$ and uncountable values of κ or λ? For example:

- *For $n \geq 2$, is it the case that $\lim^n \mathbf{A}_{\kappa,\lambda} \neq 0$ whenever $\lambda \geq \aleph_n$?*
- *For $n \geq 2$ and uncountable κ, is it the case that '$\lim^n \mathbf{A}_{\kappa,\omega} = 0$ if and only if $\lim^n \mathbf{A} = 0$'?*

References

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Thank you!