Nontrivial coherent families of functions

Lecture 2

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Review

Recall:

- For $f \in {}^\omega\omega$, $I(f) := \{(i,j) \in \omega^2 \mid j \leq f(i)\}$.
- Suppose that $\Phi = \langle \varphi_f : I(f) \to \mathbb{Z} \mid f \in {}^\omega\omega \rangle$ is a family of functions.
  - Φ is *coherent* if $\varphi_f \restriction I(f \wedge g) =^* \varphi_g \restriction I(f \wedge g)$ for all $f, g \in {}^\omega\omega$.
  - Φ is *trivial* if there is $\psi : \omega^2 \to \mathbb{Z}$ such that $\varphi_f =^* \psi \restriction I(f)$ for all $f \in {}^\omega\omega$.
- $\mathfrak{d} = \aleph_1 \Rightarrow$ there exists a nontrivial coherent family.
- After adding $\aleph_2$-many Cohen reals, every coherent family is trivial.
- $\mathrm{OCA} \Rightarrow$ every coherent family is trivial.

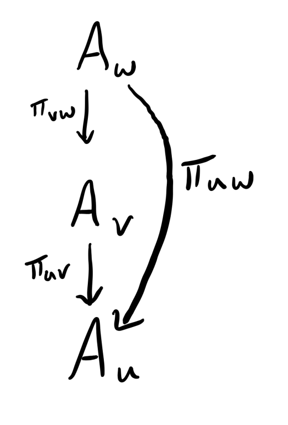
II. Homological origins

Inverse systems

Definition

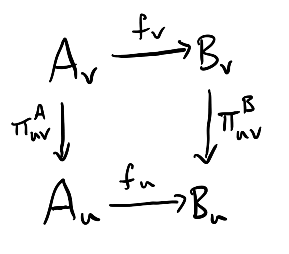
Suppose that $(\Gamma, \leq)$ is a directed set. An *inverse system* (of abelian groups) indexed by Γ is a family $\mathbf{A} = \langle A_u,\ \pi_{uv} \mid u \leq v \in \Gamma\rangle$ such that:

- for all $u \in \Gamma$, A_u is an abelian group;
- for all $u \leq v \in \Gamma$, $\pi_{uv} : A_v \to A_u$ is a group homomorphism;
- for all $u \leq v \leq w \in \Gamma$, $\pi_{uw} = \pi_{uv} \circ \pi_{vw}$.



Level morphisms

If **A** and **B** are two inverse systems indexed by the same directed set, Γ, then a *level morphism* from **A** to **B** is a family of group homomorphisms $\mathbf{f} = \langle f_u : A_u \to B_u \mid u \in \Gamma \rangle$ such that, for all $u \le v \in \Gamma$, $\pi^B_{uv} \circ f_v = f_u \circ \pi^A_{uv}$.



With this notion of morphism, the class of all inverse systems indexed by a fixed directed set Γ becomes a well-behaved category $\mathrm{Ab}^{\Gamma^{\mathrm{op}}}$ (in particular, it is an abelian category).

Inverse limits

If $\mathbf{A}$ is an inverse system indexed by Γ, then we can form the *inverse limit*, $\lim \mathbf{A}$, which is itself an abelian group. Concretely, $\lim \mathbf{A}$ can be seen as the subgroup of $\prod_{u \in \Gamma} A_u$ consisting of all sequences $\langle a_u \mid u \in \Gamma \rangle$ such that, for all $u \leq v \in \Gamma$, we have $a_u = \pi_{uv}(a_v)$.

If $\mathbf{A}$ and $\mathbf{B}$ are inverse systems and $\mathbf{f} : \mathbf{A} \to \mathbf{B}$, then $\mathbf{f}$ lifts to a group homomorphism $\lim \mathbf{f} : \lim \mathbf{A} \to \lim \mathbf{B}$. Concretely, this is done by letting $\lim \mathbf{f}(\langle a_u \mid u \in \Gamma \rangle) = \langle f_u(a_u) \mid u \in \Gamma \rangle$ for all $\langle a_u \mid u \in \Gamma \rangle \in \lim \mathbf{A}$.

This turns lim into a *functor* from the category $\mathrm{Ab}^{\Gamma^{\mathrm{op}}}$ of inverse systems indexed by Γ to the category Ab of abelian groups.

Question: How "nice" is this functor?

Exact sequences

In the category of inverse systems, kernels, images, and quotients can be defined pointwise in the obvious way. For example, if $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ is a level morphism, then $\ker(\mathbf{f})$ can be seen as the inverse system $\langle \ker(f_u), \pi_{uv} \mid u \leq v \in \Gamma \rangle$, where π_{uv} is simply $\pi^A_{uv} \restriction \ker(f_v)$.

We say that a pair of morphisms $\mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \xrightarrow{\mathbf{g}} \mathbf{C}$ is *exact at* $\mathbf{B}$ if $\mathrm{im}(\mathbf{f}) = \ker(\mathbf{g})$. A *short exact sequence* is a sequence $\mathbf{0} \to \mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \xrightarrow{\mathbf{g}} \mathbf{C} \to \mathbf{0}$ that is exact at $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

In a short exact sequence as above, we have $\ker(\mathbf{f}) = \mathbf{0}$ ($\mathbf{f}$ is *injective*) and $\mathrm{im}(\mathbf{g}) = \mathbf{C}$ ($\mathbf{g}$ is *surjective*). It can be helpful to think of $\mathbf{A}$ as a *subobject* of $\mathbf{B}$ and to think of $\mathbf{C}$ as the quotient $\mathbf{B}/\mathbf{A}$.

Exact functors

A functor F between abelian categories is said to be *exact* if it preserves short exact sequences, i.e., if, whenever $\mathbf{0} \to \mathsf{A} \xrightarrow{\mathbf{f}} \mathsf{B} \xrightarrow{\mathbf{g}} \mathsf{C} \to \mathbf{0}$ is exact in the source category of F, $\mathbf{0} \to F\mathsf{A} \xrightarrow{F\mathbf{f}} F\mathsf{B} \xrightarrow{F\mathbf{g}} F\mathsf{C} \to \mathbf{0}$ is exact in the target category of F.

The inverse limit functor is *left exact*: if $\mathbf{0} \to \mathsf{A} \xrightarrow{\mathbf{f}} \mathsf{B} \xrightarrow{\mathbf{g}} \mathsf{C}$ is exact at A and B, then $\mathbf{0} \to \lim \mathsf{A} \xrightarrow{\lim \mathbf{f}} \lim \mathsf{B} \xrightarrow{\lim \mathbf{g}} \lim \mathsf{C}$ is exact at $\lim \mathsf{A}$ and $\lim \mathsf{B}$. However, it fails to be exact, i.e., even if $\mathrm{im}(\mathbf{g}) = \mathsf{C}$, we might have $\mathrm{im}(\lim \mathbf{g}) \neq \lim \mathsf{C}$.

The failure of lim to be exact essentially amounts to the failure of lim to preserve quotients: if the quotient system B/A is defined, then it need not be the case that $\lim \mathsf{B}/\mathsf{A} \cong \lim \mathsf{B}/\lim \mathsf{A}$.

An example ($\Gamma = \omega$)

$$\mathbf{0} \longrightarrow \mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \xrightarrow{\mathbf{g}} \mathbf{C} \longrightarrow \mathbf{0}$$

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 3} & \mathbb{Z} & \xrightarrow{\text{mod } 3} & \mathbb{Z}/3 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 3} & \mathbb{Z} & \xrightarrow{\text{mod } 3} & \mathbb{Z}/3 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 3} & \mathbb{Z} & \xrightarrow{\text{mod } 3} & \mathbb{Z}/3 & \longrightarrow & 0
\end{array}$$

$\lim \mathbf{A} = \lim \mathbf{B} = 0$ and $\lim \mathbf{C} = \mathbb{Z}/3$, so applying lim to this short exact sequence yields $0 \to 0 \to 0 \to \mathbb{Z}/3 \to 0$, which is not exact at $\mathbb{Z}/3$.

Derived functors

Given any left exact functor F, there is a general procedure for producing a sequence of (right) derived functors $\langle F^n \mid n \in \omega \setminus \{0\}\rangle$ that "measure" the failure of the functor F to be exact. These derived functors then take short exact sequences

$$\mathbf{0} \longrightarrow \mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \xrightarrow{\mathbf{g}} \mathbf{C} \longrightarrow \mathbf{0}$$

to *long* exact sequences

$$\begin{array}{l} \mathbf{0} \longrightarrow F\mathbf{A} \xrightarrow{F\mathbf{f}} F\mathbf{B} \xrightarrow{F\mathbf{g}} F\mathbf{C} \xrightarrow{\delta} \\ F^1\mathbf{A} \xrightarrow{F^1\mathbf{f}} F^1\mathbf{B} \xrightarrow{F^1\mathbf{g}} F^1\mathbf{C} \xrightarrow{\delta} \\ F^2\mathbf{A} \xrightarrow{F^2\mathbf{f}} F^2\mathbf{B} \xrightarrow{F^2\mathbf{g}} F^2\mathbf{C} \longrightarrow \ldots \end{array}$$

We will be interested in the derived functors $\langle \lim^n \mid n \in \omega \setminus \{0\}\rangle$.

The system $\mathbf{A}$

Consider the directed set $({}^{\omega}\omega, \leq)$, and define an inverse system $\mathbf{A} = \langle A_f,\ \pi_{fg} \mid f \leq g \in {}^{\omega}\omega \rangle$ as follows:

- $A_f = \bigoplus_{I(f)} \mathbb{Z}$
- $\pi_{fg} : A_g \to A_f$ is the natural projection map.

In other words, A_g is the group of finitely supported functions $\varphi : I(g) \to \mathbb{Z}$, and, if $f \leq g$, then π_{fg} takes such a function φ to $\varphi \restriction I(f)$.

Question: What is $\lim \mathbf{A}$?
Answer: $\lim \mathbf{A} \cong \bigoplus_{\omega} \prod_{\omega} \mathbb{Z}$.

Define $\mathbf{B} = \langle B_f,\ \pi_{fg} \mid f \leq g \in {}^{\omega}\omega \rangle$ similarly by letting $B_f = \prod_{I(f)} \mathbb{Z}$.

Question: What is $\lim \mathbf{B}$?
Answer: $\lim \mathbf{B} \cong \prod_{\omega^2} \mathbb{Z}$.

$\lim^1 \mathbf{A}$

There is a natural inclusion morphism $\mathbf{i} : \mathbf{A} \to \mathbf{B}$ and a quotient morphism $\mathbf{q} : \mathbf{B} \to \mathbf{B}/\mathbf{A} = \langle B_f/A_f,\ \pi_{fg} \mid f \leq g \in {}^\omega\omega \rangle$. Then the short exact sequence

$$\mathbf{0} \longrightarrow \mathbf{A} \xrightarrow{\mathbf{i}} \mathbf{B} \xrightarrow{\mathbf{q}} \mathbf{B}/\mathbf{A} \longrightarrow \mathbf{0}$$

gives rise to the long exact sequence

$$0 \longrightarrow \lim \mathbf{A} \xrightarrow{\lim \mathbf{i}} \lim \mathbf{B} \xrightarrow{\lim \mathbf{q}} \lim \mathbf{B}/\mathbf{A} \xrightarrow{\delta} \lim^1 \mathbf{A} \xrightarrow{\lim^1 \mathbf{i}} \lim^1 \mathbf{B} \xrightarrow{\lim^1 \mathbf{q}} \lim^1 \mathbf{B}/\mathbf{A} \xrightarrow{\delta} \lim^2 \mathbf{A} \xrightarrow{\lim^2 \mathbf{i}} \lim^2 \mathbf{B} \xrightarrow{\lim^2 \mathbf{q}} \lim^2 \mathbf{B}/\mathbf{A} \longrightarrow \dots$$

$\lim^1 \mathbf{A}$

It can be shown that $\lim^n \mathbf{B} = 0$ for all $n \in \omega \setminus \{0\}$, so this long exact sequence becomes

$$0 \longrightarrow \lim \mathbf{A} \xrightarrow{\lim \iota} \lim \mathbf{B} \xrightarrow{\lim \mathbf{q}} \lim \mathbf{B}/\mathbf{A} \xrightarrow{\delta} \lim^1 \mathbf{A} \longrightarrow 0 \longrightarrow \lim^1 \mathbf{B}/\mathbf{A} \xrightarrow{\delta} \lim^2 \mathbf{A} \longrightarrow 0 \longrightarrow \lim^2 \mathbf{B}/\mathbf{A} \longrightarrow \dots$$

It follows that $\lim^1 \mathbf{A} \cong \frac{\lim \mathbf{B}/\mathbf{A}}{\mathrm{im}(\lim \mathbf{q})}$ and, for $n \geq 1$, $\lim^{n+1} \mathbf{A} \cong \lim^n(\mathbf{B}/\mathbf{A})$.

$\lim^1 \mathbf{A}$

We have $\lim^1 \mathbf{A} \cong \frac{\lim \mathbf{B}/\mathbf{A}}{\mathrm{im}(\lim \mathbf{q})}$.

What is $\lim \mathbf{B}/\mathbf{A}$? Recall that

$$\mathbf{B}/\mathbf{A} = \left\langle \prod_{I(f)} \mathbb{Z} / \bigoplus_{I(f)} \mathbb{Z},\ \pi_{fg} \;\middle|\; f \leq g \in {}^{\omega}\omega \right\rangle.$$

$\lim \mathbf{B}/\mathbf{A}$ therefore consists of families $\langle [\varphi_f] \mid f \in {}^{\omega}\omega \rangle$ such that

- $\varphi_f \in B_f$, i.e., $\varphi_f : I(f) \to \mathbb{Z}$;
- $[\varphi_f]$ is the equivalence class of all functions $\varphi : I(f) \to \mathbb{Z}$ for which $\varphi_f - \varphi \in A_f$, i.e., all functions $\varphi : I(f) \to \mathbb{Z}$ that differ from φ_f in only finitely many places;
- for all $f \leq g \in {}^{\omega}\omega$, we have $[\varphi_f] = [\pi_{fg}\varphi_g] = [\varphi_g \restriction I(f)]$, i.e., $\varphi_f =^* \varphi_g \restriction I(f)$.

Thus, $\lim \mathbf{B}/\mathbf{A}$ consists precisely of (equivalence classes of) coherent families of functions!

$\lim^1 \mathbf{A}$

We have $\lim^1 \mathbf{A} \cong \frac{\lim \mathbf{B}/\mathbf{A}}{\mathrm{im}(\lim \mathbf{q})}$ and $\lim \mathbf{B}/\mathbf{A}$ consists of equivalence classes of coherent families of functions.

What is $\mathrm{im}(\lim \mathbf{q})$? $\lim \mathbf{q} : \lim \mathbf{B} \to \lim \mathbf{B}/\mathbf{A}$. Recall that $\lim \mathbf{B} \cong \prod_{\omega^2} \mathbb{Z}$, so $\lim \mathbf{B}$ can be thought of as the set of all $\psi : \omega^2 \to \mathbb{Z}$. Such a function ψ gets sent by $\lim \mathbf{q}$ to $\langle [\psi \restriction I(f)] \mid f \in {}^\omega\omega \rangle$, which is (the equivalence class of) a coherent family that is *trivial*, as witnessed by ψ.

Also, (the equivalence class of) *every* trivial coherent family lies in $\mathrm{im}(\lim \mathbf{q})$: if $\langle \varphi_f \mid f \in {}^\omega\omega \rangle$ is trivialized by $\psi : \omega^2 \to \mathbb{Z}$, then $\langle [\varphi_f] \mid f \in {}^\omega\omega \rangle = \lim \mathbf{q}(\psi)$.

So we can think of $\lim^1 \mathbf{A}$ as $\frac{\text{coherent families of functions}}{\text{trivial coherent families of functions}}$. In particular, $\lim^1 \mathbf{A} = 0$ if and only if every coherent family of functions is trivial.

2-dimensional nontrivial coherence

Definition

Let $\Phi = \langle \varphi_{fg} : I(f \wedge g) \to \mathbb{Z} \mid f, g \in {}^{\omega}\omega \rangle$.

1. Φ is *alternating* if $\varphi_{fg} = -\varphi_{gf}$ for all $f, g \in {}^{\omega}\omega$.
2. Φ is *2-coherent* if it is alternating and $\varphi_{fg} + \varphi_{gh} =^* \varphi_{fh}$ for all $f, g, h \in {}^{\omega}\omega$. (All functions restricted to $I(f \wedge g \wedge h)$.)
3. Φ is *2-trivial* if there is a family

$$\Psi = \langle \psi_f : I(f) \to \mathbb{Z} \mid f \in {}^{\omega}\omega \rangle$$

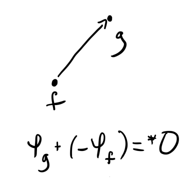
such that $\psi_g - \psi_f =^* \varphi_{fg}$ for all $f, g \in {}^{\omega}\omega$.

A 2-trivial family is 2-coherent. A non-2-trivial 2-coherent family Φ is an example of incompactness: each *local family* $\{\varphi_{fg} \mid f, g < h\}$ (for a fixed $h \in {}^{\omega}\omega$) is 2-trivial, as witnessed by the family $\langle -\varphi_{fh} \mid f < h \rangle$, but the entire family is not.

A reframing

Coherence and triviality can be reframed in terms of oriented sums of functions indexed by maximal faces of simplices whose vertices are elements of ${}^{\omega}\omega$. For a finite sequence $\vec{f} = \langle f_0, \ldots, f_{n-1} \rangle$, let $I(\vec{f})$ denote $I(f_0) \cap \ldots \cap I(f_{n-1})$. In particular, we let $I(\emptyset) = \omega^2$.

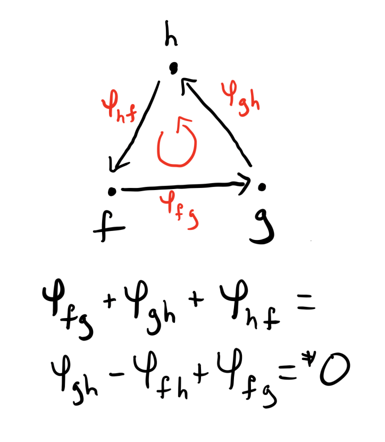
A 1-dimensional family $\Phi = \langle \varphi_f \mid f \in {}^{\omega}\omega \rangle$ is *coherent* if the oriented sum on the boundary of every 1-simplex vanishes mod finite:



It is *trivial* if the information in the 1-dimensional family Φ is contained (mod finite) in a 0-dimensional family $\langle \psi_\emptyset : I(\emptyset) \to \mathbb{Z} \rangle$.

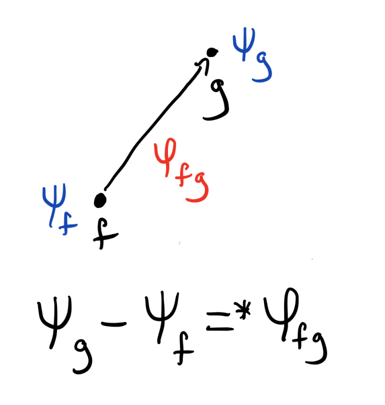
A reframing

A 2-dimensional family $\Phi = \langle \varphi_{fg} : I(f,g) \to \mathbb{Z} \mid f, g \in {}^\omega\omega \rangle$ is *coherent* if the oriented sum on the boundary of every 2-simplex vanishes mod finite:



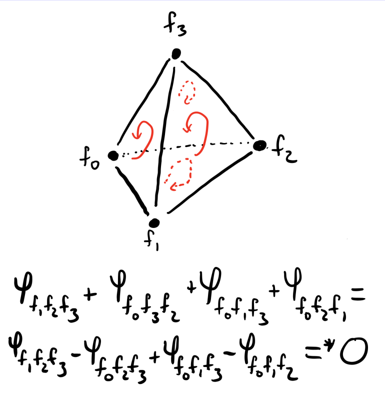
A reframing

A 2-dimensional family $\Phi = \langle \varphi_{fg} : I(f,g) \to \mathbb{Z} \mid f,g \in {}^{\omega}\omega \rangle$ is *trivial* if the information in the 2-dimensional family Φ is contained (mod finite) in a 1-dimensional family $\langle \psi_f : I(f) \to \mathbb{Z} \mid f \in {}^{\omega}\omega \rangle$:



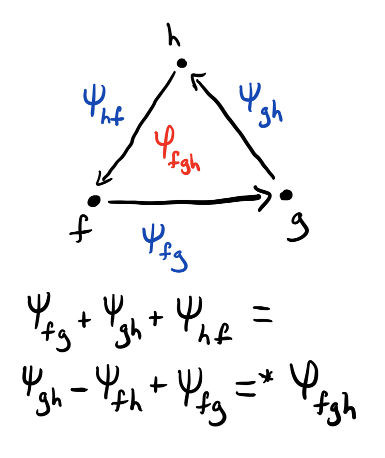
A reframing

A 3-dimensional family $\Phi = \langle \varphi_{fgh} : I(f,g,h) \to \mathbb{Z} \mid f,g,h \in {}^{\omega}\omega \rangle$ is *coherent* if the oriented sum on the boundary of every 3-simplex vanishes mod finite:



A reframing

A 3-dimensional family $\Phi = \langle \varphi_{fgh} : I(f,g,h) \to \mathbb{Z} \mid f,g,h \in {}^{\omega}\omega \rangle$ is *trivial* if the information in the 3-dimensional family Φ is contained (mod finite) in a 2-dimensional family $\langle \psi_{fg} \mid f,g \in {}^{\omega}\omega \rangle$:



n-dimensional nontrivial coherence

Given a sequence $\vec{f} = (f_0, \ldots, f_{n-1})$ and $i < n$, $\vec{f}^i$ is the sequence of length $n-1$ formed by removing f_i from $\vec{f}$.

Definition

Fix $n \geq 2$, and let $\Phi = \left\langle \varphi_{\vec{f}} : I(\wedge \vec{f}) \to \mathbb{Z} \;\middle|\; \vec{f} \in ({}^\omega\omega)^n \right\rangle$.

1. Φ is *alternating* if $\varphi_{\vec{f}} = sgn(\sigma)\varphi_{\sigma(\vec{f})}$ for all $\vec{f} \in ({}^\omega\omega)^n$ and all permutations σ.
2. Φ is *n-coherent* if it is alternating and $\sum_{i=0}^{n}(-1)^i \varphi_{\vec{f}^i} =^* 0$ for all $\vec{f} \in ({}^\omega\omega)^{n+1}$.
3. Φ is *n-trivial* if there is an alternating family

$$\left\langle \psi_{\vec{f}} : I(\wedge \vec{f}) \to \mathbb{Z} \;\middle|\; \vec{f} \in ({}^\omega\omega)^{n-1} \right\rangle$$

such that $\sum_{i=0}^{n-1}(-1)^i \psi_{\vec{f}^i} =^* \varphi_{\vec{f}}$ for all $\vec{f} \in ({}^\omega\omega)^n$.

$\lim^n \mathbf{A}$ and nontrivial coherence

Coherent and *trivial* can now be thought of as 1-coherent and 1-trivial.

Theorem (Mardešić-Prasolov ($n = 1$), '88, [3], Bergfalk ($n \geq 2$), '17, [1])

Fix $n \geq 1$. Then $\lim^n \mathbf{A} = 0$ if and only if every n-coherent family

$$\Phi = \left\langle \varphi_{\vec{f}} \;\middle|\; \vec{f} \in ({}^{\omega}\omega)^n \right\rangle$$

is n-trivial.

Thus, to prove that $\lim^n \mathbf{A} = 0$, it suffices to show that every n-coherent family is n-trivial.

Additivity of homology

Definition

A homology theory is *additive* on a class of topological spaces $\mathcal{C}$ if, for every natural number p and every family $\{X_i \mid i \in J\}$ such that each X_i and $\coprod_J X_i$ are in $\mathcal{C}$, we have

$$\bigoplus_J \mathrm{H}_p(X_i) \cong \mathrm{H}_p(\coprod_J X_i)$$

via the map induced by the inclusions

$$X_i \hookrightarrow \coprod_J X_i.$$

Additivity of strong homology

Let X^n denote the n-dimensional infinite earring space, i.e., the one-point compactification of an infinite countable sum of copies of the n-dimensional open unit ball. Let $\bar{\mathrm{H}}_p(X)$ denote the p^{th} strong homology group of X.

Theorem (Mardešić-Prasolov, '88, [3])

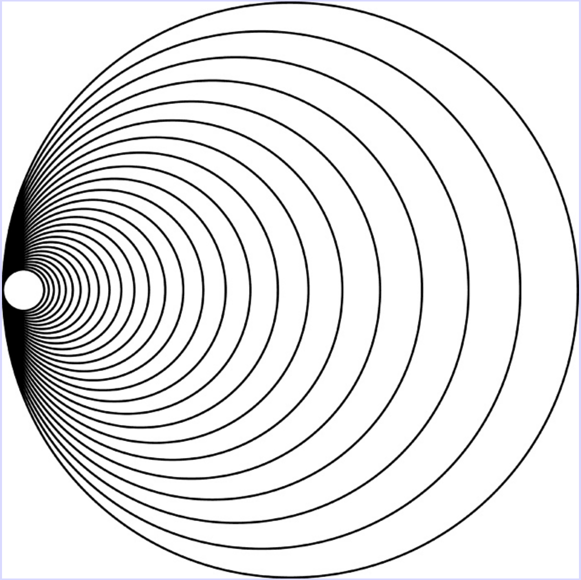
Suppose that $0 < p < n$ are natural numbers. Then

$$\bigoplus_{\mathbb{N}} \bar{\mathrm{H}}_p(X^n) = \bar{\mathrm{H}}_p(\coprod_{\mathbb{N}} X^n)$$

if and only if $\lim^{n-p} \mathbf{A} = 0$.

Consequently, if strong homology is additive on closed subsets of Euclidean space, then $\lim^n \mathbf{A} = 0$ for all $n \geq 1$.

An infinite earring (X^1)



Condensed mathematics

The question of the consistency of $\lim^n \mathbf{A} = 0$ for $n \geq 1$ arose independently in recent work of Clausen and Scholze on *condensed mathematics*, a new approach to doing algebra in situations in which the algebraic structures carry topologies. They introduce the category of *condensed abelian groups*, which is a much nicer category algebraically than the category of *topological abelian groups*. The natural question of whether pro-abelian groups embed fully faithfully into condensed abelian groups is equivalent, in its simplest case, to the question of whether $\lim^n \mathbf{A} = 0$ for $n \geq 1$.

Looking ahead

We are now interested in the following general questions:

Question

What can be said about the conditions under which $\lim^n \mathbf{A} = 0$ *for* $n > 1$.

Question

In particular, is it consistent that $\lim^n \mathbf{A} = 0$ *simultaneously for all* $n \geq 1$?

Question

What happens with $\lim^n \mathbf{A}^*$ *for other natural inverse systems* $\mathbf{A}^*$?

Some partial answers will come in the next lecture.

References

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