# Nontrivial coherent families of functions

## Lecture 1

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# Introduction

# Nontrivial coherence

This tutorial will be about set theoretic combinatorial objects that are

1. Coherent (i.e., *locally* trivial)
2. Nontrivial (i.e., not *globally* trivial)

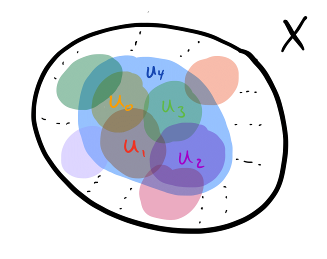
Such objects are thus naturally viewed as instances of set theoretic *incompactness*, in which a structure's global properties differ markedly from its local properties.

The study of set theoretic incompactness has been a main current of infinitary combinatorics research for the last 50+ years, with the following general heuristic emerging:

- Instances of *incompactness* abound in canonical inner models such as Gödel's L (square principles, higher Suslin trees, ...).
- Instances of *compactness* follow from forcing axioms or the existence of large cardinals.

# An archetypal example

Suppose that $X$ and $Y$ are sets and $X = \bigcup_{i \in I} U_i$. (Think of the $U_i$'s as being "small" subsets of $X$.)




Let $\Phi = \langle \varphi_i : U_i \to Y \mid i \in I \rangle$ be a family of functions.

- $\Phi$ is *coherent* if, for all $i, j \in I$, $\varphi_i \restriction (U_i \cap U_j) =^* \varphi_j \restriction (U_i \cap U_j)$.
- $\Phi$ is *trivial* if there is $\psi : X \to Y$ such that $\psi \restriction U_i =^* \varphi_i$ for all $i \in I$.

Note that coherence can be seen as *local triviality* : for all $j \in I$, the family of functions $\Phi \restriction U_j := \langle \varphi_i \mid i \in I \text{ and } U_i \subseteq U_j \rangle$ is trivial, as witnessed by $\varphi_j$.

# Outline of tutorial

This tutorial will focus on nontrivial coherent families of functions indexed by (finite powers of) ${}^\omega\omega$. Such families are natural set theoretic objects in their own right but also arise from work in other fields, notably homological algebra. We will focus mainly on the set theoretic aspects of their study but will also take some time to note their connections to and implications for other fields.

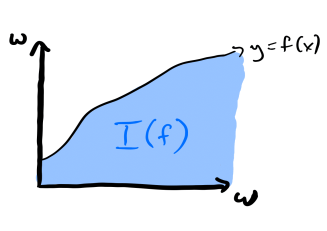
**Lecture 1:** Set theoretic aspects of 1-dimensional nontrivial coherent families

**Lecture 2:** Homological origins, and an introduction to higher dimensional nontrivial coherence

**Lecture 3:** Recent results on higher dimensional nontrivial coherence

# I. 1-coherence and triviality

# Basic definitions




Given a function $f \in {}^{\omega}\omega$, let $I(f) := \{(i,j) \in \omega^2 \mid j \leq f(i)\}$, Visually, $I(f)$ is simply the region below (or on) the graph of $f$.

Let $I^+(f) := \{(i, f(i)) \mid i < \omega\}$, and $I^-(f) := I(f) \setminus I^+(f)$.

Given functions $f, g \in {}^{\omega}\omega$:

- $f \leq g$ if $f(i) \leq g(i)$ for all $i < \omega$;
- $f \leq^* g$ if $f(i) \leq g(i)$ for all but finitely many $i < \omega$;
- $f \wedge g$ is the pointwise minimum of $f$ and $g$;
- if $\varphi$ and $\psi$ are functions, we write $\varphi =^* \psi$ to assert that $\varphi(a) = \psi(a)$ for all but finitely many $a \in \mathrm{dom}(\varphi) \cap \mathrm{dom}(\psi)$.

# Coherence and triviality

**Definition**

Suppose that $\Phi = \langle \varphi_f : I(f) \to \mathbb{Z} \mid f \in {}^\omega\omega \rangle$ is a family of functions.

1. $\Phi$ is *coherent* if $\varphi_f =^* \varphi_g$ for all $f, g \in {}^\omega\omega$.
2. $\Phi$ is *trivial* if there is $\psi : \omega^2 \to \mathbb{Z}$ such that $\psi =^* \varphi_f$ for all $f \in {}^\omega\omega$. (We say that $\psi$ *trivializes* $\Phi$.)

**Question:** Do nontrivial coherent families exist?

A preliminary observation: The definitions of coherent and trivial also make sense for *partial* families $\Phi_{\mathcal{F}} = \langle \varphi_f \mid f \in \mathcal{F} \rangle$, where $\mathcal{F} \subseteq {}^\omega\omega$. If $\mathcal{F}$ is a $\leq^*$-cofinal subset of ${}^\omega\omega$, then such a family $\Phi_{\mathcal{F}}$ can be extended to a total family $\Phi$ as follows: given $g \in {}^\omega\omega \setminus \mathcal{F}$, find $f \in \mathcal{F}$ such that $g \leq^* f$ and let $\varphi_g =^* \varphi_f \restriction I(g)$. Then this family $\Phi$ is trivial if and only if $\Phi_{\mathcal{F}}$ is trivial.

# Two basic lemmata

**Lemma**

*Suppose that $\mathcal{F} \subseteq {}^{\omega}\omega$ and $\Phi = \langle \varphi_f \mid f \in \mathcal{F} \rangle$ is coherent. Then the following are equivalent:*

1. *$\Phi$ is trivial;*
2. *there is a family $\langle \psi_f : I(f) \to \mathbb{Z} \mid f \in \mathcal{F} \rangle$ of finitely supported functions such that, for all $f, g \in \mathcal{F}$,*

$$\varphi_f - \psi_f = \varphi_g - \psi_g \quad (\textit{on } I(f \wedge g)).$$

**Proof sketch.**

$1 \Rightarrow 2$: If $\psi : \omega^2 \to \mathbb{Z}$ trivializes $\Phi$, then let $\psi_f = \varphi_f - \psi \restriction I(f)$ for all $f \in \mathcal{F}$.

$2 \Rightarrow 1$: Given $\langle \psi_f \mid f \in \mathcal{F} \rangle$, let $\psi = \bigcup_{f \in \mathcal{F}} (\varphi_f - \psi_f)$ (extend arbitrarily to cover all of $\omega^2$ if necessary.) Then $\psi$ trivializes $\Phi$. $\square$

## Lemma

*Suppose that* $\mathcal{F} \subseteq {}^{\omega}\omega$ *is countable and* $\Phi = \langle \varphi_f \mid f \in \mathcal{F} \rangle$ *is coherent. Then* $\Phi$ *is trivial.*

## Proof.

Enumerate $\mathcal{F}$ as $\langle f_n \mid n < \omega \rangle$. Define a function $\psi : \omega^2 \to \mathbb{Z}$ as follows: for all $(i, j) \in \omega^2$, let $\psi(i, j) = \varphi_{f_n}(i, j)$, where $n < \omega$ is least such that $(i, j) \in I(f_n)$ (if such an $n$ exists).

We claim that $\psi$ trivializes $\Phi$. If not, then there is $n < \omega$ and an infinite $E \subseteq I(f_n)$ such that $\psi(i, j) \neq \varphi_{f_n}(i, j)$ for all $(i, j) \in E$. For each $(i, j) \in E$, there is $m \leq n$ such that $\psi(i, j) = \varphi_{f_m}(i, j)$. There is therefore an infinite $E' \subseteq E$ and an $m \leq n$ such that $\psi(i, j) = \varphi_m(i, j)$ for all $(i, j) \in E'$. But then, for all $(i, j) \in E'$, we have

$$\varphi_{f_m}(i, j) = \psi(i, j) \neq \varphi_{f_n}(i, j),$$

so $\neg(\varphi_{f_m} =^* \varphi_{f_n})$, contradicting the fact that $\Phi$ is coherent. $\square$

# CH and nontrivial coherence

**Theorem (Mardešić-Prasolov, Simon, '88, [3])**

$\mathrm{CH} \Rightarrow$ *there exists a nontrivial coherent family.*

**Theorem (Dow-Simon-Vaughan, '89, [1])**

$\mathfrak{d} = \aleph_1 \Rightarrow$ *there exists a nontrivial coherent family.*

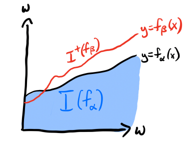
**Proof.**

Let $\vec{f} = \langle f_\alpha \mid \alpha < \omega_1 \rangle$ be $<^*$-increasing and $<^*$-cofinal in ${}^\omega\omega$. By recursion on $\alpha$, we will define a coherent (partial) family $\langle \varphi_\alpha : I(f_\alpha) \to \omega \mid \alpha < \omega_1 \rangle$ and then will prove that it is nontrivial. Since $\vec{f}$ is $<^*$-cofinal, we can then extend this family to a nontrivial coherent family defined on all of ${}^\omega\omega$.

At the start, fix a bijection $\pi : \omega^2 \to \omega$.

## Proof (cont.)

Suppose that $\beta < \omega_1$ and we have defined $\langle \varphi_\alpha \mid \alpha < \beta \rangle$. Since $\beta$ is countable, $\langle \varphi_\alpha \mid \alpha < \beta \rangle$ is trivial, so we can find $\varphi : I(f_\beta) \to \omega$ such that $\varphi =^* \varphi_\alpha$ for all $\alpha < \beta$.
For all $\alpha < \beta$, since $f_\alpha <^* f_\beta$, we have $|I^+(f_\beta) \cap I(f_\alpha)| < \aleph_0$.




Therefore, we can adjust the values of $\varphi$ on $I^+(f_\beta)$ while maintaining the coherence with $\langle \varphi_\alpha \mid \alpha < \beta \rangle$. With this in mind, define $\varphi_\beta$ as follows:

- $\varphi_\beta \restriction I^-(f_\beta) = \varphi \restriction I^-(f_\beta)$;
- for all $i < \omega$, $\varphi_\beta(i, f_\beta(i)) = f_\beta(\pi(i, f_\beta(i)))$.

□

## Proof (conclusion).

For all $\alpha < \omega_1$ and $i < \omega$, $\varphi_\alpha(i, f_\alpha(i)) = f_\alpha(\pi(i, f_\alpha(i)))$.

**Claim:** $\langle \varphi_\alpha \mid \alpha < \omega_1 \rangle$ is nontrivial.
Suppose for sake of contradiction that $\psi : \omega^2 \to \omega$ and $\psi \restriction I(f_\alpha) =^* \varphi_\alpha$ for all $\alpha < \omega_1$. Find $\alpha < \omega_1$ such that $\psi \circ \pi^{-1} <^* f_\alpha$. Then, for all but finitely many $i < \omega$, we have

$$\begin{aligned} \varphi_\alpha(i, f_\alpha(i)) &= \psi(i, f_\alpha(i)) \\ &= \psi \circ \pi^{-1}(\pi(i, f_\alpha(i))) \\ &< f_\alpha(\pi(i, f_\alpha(i))) \\ &= \varphi_\alpha(i, f_\alpha(i)), \end{aligned}$$

which is a contradiction. Therefore, $\langle \varphi_\alpha \mid \alpha < \omega_1 \rangle$ is nontrivial. □

# Cohen forcing

Theorem (Kamo, '93, [2])

*Let $\mathbb{P}$ be the forcing to add $\aleph_2$-many Cohen reals. Then, in $V^{\mathbb{P}}$, every coherent family of functions is trivial.*

Idea of the proof.

- If $\Phi = \langle \varphi_f \mid f \in {}^\omega\omega \rangle$ is nontrivial and coherent and $\mathcal{F} \subseteq {}^\omega\omega$ is $<^*$-unbounded and consists of increasing functions, then $\Phi \restriction \mathcal{F} = \langle \varphi_f \mid f \in \mathcal{F} \rangle$ is nontrivial.
- Suppose that $\mathbb{P}$ adds Cohen reals $\langle \dot{f}_\alpha \mid \alpha < \omega_2 \rangle$, and assume that each $\dot{f}_\alpha$ is an increasing element of ${}^\omega\omega$.
- Suppose that $\dot{\Phi} = \langle \dot{\varphi}_{\dot{f}} \mid \dot{f} \in {}^\omega\omega \rangle$ is a name for a nontrivial coherent family.
- $\langle \dot{f}_\alpha \mid \alpha < \omega_1 \rangle$ is forced to be $<^*$-unbounded, so $\langle \dot{\varphi}_{\dot{f}_\alpha} \mid \alpha < \omega_1 \rangle$ is forced to be nontrivial.

Idea of the proof (cont.).

- By the chain condition of $\mathbb{P}$, there is some $\beta < \omega_2$ such that $\langle \dot{\varphi}_{\dot{f}_\alpha} \mid \alpha < \omega_1 \rangle$ is forced to be in the intermediate model obtained just by adding the first $\beta$-many Cohen reals, and it is nontrivial there.
- The nontriviality of $\langle \varphi_{f_\alpha} \mid \alpha < \omega_1 \rangle$ prevents it from being extended to incorporate any of the remaining Cohen reals yet to be added (for example, $\dot{f}_\beta$).

□

# The Open Coloring Axiom

**Definition (Todorcevic)**

The *Open Coloring Axiom* ($\mathrm{OCA}$) is the following assertion: For every separable metric space $X$ and every partition

$$[X]^2 = K_0 \cup K_1$$

such that $K_0$ is open in $[X]^2$, one of the following holds:

1. there is an uncountable $Y \subseteq X$ such that $[Y]^2 \subseteq K_0$; or
2. there is a countable partition of $X$, $X = \bigcup_{n<\omega} X_n$, such that, for all $n < \omega$, $[X_n]^2 \subseteq K_1$.

- The Proper Forcing Axiom ($\mathrm{PFA}$) implies $\mathrm{OCA}$.
- If $\mathrm{CH}$ holds, then there is a ccc poset that forces $\mathrm{OCA}$.
- $\mathrm{OCA} \Rightarrow \mathfrak{b} > \aleph_1$.

# OCA and nontrivial coherence

**Theorem (Dow-Simon-Vaughan '89, [1])**

$\mathrm{PFA} \Rightarrow$ *every coherent family is trivial.*

**Theorem (Todorcevic '89, [4])**

$\mathrm{OCA} \Rightarrow$ *every coherent family is trivial.*

**Proof.**

Assume $\mathrm{OCA}$, and let $\Phi = \langle \varphi_f : f \in {}^{\omega}\omega \rangle$ be a coherent family. We will show that $\Phi$ is trivial. For distinct $f, g \in {}^{\omega}\omega$, let $\Delta(f, g)$ be the least $i < \omega$ such that either

- $f(i) \neq g(i)$; or
- there is $j$ such that $(i, j) \in I(f \wedge g)$ and $\varphi_f(i, j) \neq \varphi_g(i, j)$.

Set $d(f, g) = 1/2^{\Delta(f,g)}$. Then $({}^{\omega}\omega, d)$ is a separable metric space.

## Proof (cont.)

Let $K_1$ be the set of $\{f, g\} \in [{}^\omega\omega]^2$ such that $\varphi_f \restriction I(f \wedge g) = \varphi_g \restriction I(f \wedge g)$, and let $K_0 = [{}^\omega\omega]^2 \setminus K_1$.

**Claim:** $K_0$ is open in $[{}^\omega\omega]^2$ (with respect to $d$).

**Proof of claim:** Suppose that $\{f, g\} \in K_0$, and let $(i, j) \in I(f \wedge g)$ be such that $\varphi_f(i, j) \neq \varphi_g(i, j)$. Then, if $f'$ and $g'$ are such that $\max\{d(f, f'), d(g, g')\} < 1/2^i$, we have $(i, j) \in I(f' \wedge g')$ and

$$\varphi_{f'}(i, j) = \varphi_f(i, j) \neq \varphi_g(i, j) = \varphi_{g'}(i, j),$$

so $\{f', g'\} \in K_0$.

We now apply $\mathrm{OCA}$ to the partition $[{}^\omega\omega]^2 = K_0 \cup K_1$, and consider the two possible outcomes in turn. □

## Proof (cont.)

Suppose first that there is an uncountable $Y \subseteq {}^{\omega}\omega$ such that $[Y]^2 \subseteq K_0$, i.e., for all distinct $f, g \in Y$, there is $(i,j) \in I(f \wedge g)$ such that $\varphi_f(i,j) \neq \varphi_g(i,j)$. WLOG, assume that $|Y| = \aleph_1$. Since $\mathfrak{b} > \aleph_1$, we can find $h \in {}^{\omega}\omega$ such that $f <^* h$ for all $f \in Y$.

For each $f \in Y$, let $e_f$ be the set of all $(i,j) \in I(f)$ such that either

- $(i,j) \notin I(h)$; or
- $(i,j) \in I(h)$ but $\varphi_f(i,j) \neq \varphi_h(i,j)$.

By the coherence of $\Phi$ and the fact that $f <^* h$, we know that $e_f$ is finite. Therefore, since $Y$ is uncountable, we can find distinct $f, g \in Y$ such that $e_f = e_g$ and $\varphi_f \restriction e_f = \varphi_g \restriction e_g$. But then $\varphi_f \restriction I(f \wedge g) = \varphi_g \restriction I(f \wedge g)$, so $\{f, g\} \notin K_0$. This is a contradiction to the assumption that $[Y]^2 \subseteq K_0$.

## Proof (conclusion).

Therefore, there is a countable partition ${}^{\omega}\omega = \bigcup_{n<\omega} X_n$ such that, for all $n < \omega$, $[X_n]^2 \subseteq K_1$, i.e., for all $f, g \in X_n$, $\varphi_f \restriction I(f \wedge g) = \varphi_g \restriction I(f \wedge g)$. Find $n < \omega$ such that $X_n$ is $<^*$-cofinal in ${}^{\omega}\omega$. Then let $\psi = \bigcup_{f \in X_n} \varphi_f$ (extend $\psi$ to all of $\omega^2$ arbitrarily, if necessary). Now check that $\psi \restriction I(g) =^* \varphi_g$ for all $g \in {}^{\omega}\omega$, so $\Phi$ is in fact trivial. □

**Remark:** $\mathrm{OCA}$ is true *in ZFC* if the space $X$ under consideration is an analytic subset of a Polish space. Also, there is a natural way to code a coherent family of functions as a subset $X$ of the irrationals. This proof then shows that, if the family is nontrivial, then $X$ is not analytic. On the other hand, the coherence of the family implies that $X \cap K$ is $F_\sigma$ for every compact subset $K$ of the irrationals.

# References


- Alan Dow, Petr Simon, and Jerry E. Vaughan, *Strong homology and the proper forcing axiom*, Proc. Amer. Math. Soc. **106** (1989), no. 3, 821–828.
- Shizuo Kamo, *Almost coinciding families and gaps in $\mathscr{P}(\omega)$*, J. Math. Soc. Japan **45** (1993), no. 2, 357–368.
- S. Mardešić and A. V. Prasolov, *Strong homology is not additive*, Trans. Amer. Math. Soc. **307** (1988), no. 2, 725–744.
- Stevo Todorčević, *Partition problems in topology*, Contemporary Mathematics, vol. 84, American Mathematical Society, Providence, RI, 1989. MR 980949