Borel and Borel* sets in generalized descriptive set theory

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Classical descriptive set theory

Descriptive set theory is the study of "definable sets" in Polish spaces.

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A crucial example

The Cantor space ${}^{\omega}2$ and the Baire space ${}^{\omega}\omega$. Let $A \in \{2, \omega\}$, we equip ${}^{\omega}A = \{f \mid f : \omega \to A\}$ with the topology generated by the sets

$$N_s({}^{\omega}A) := \left\{ x \in {}^{\omega}A \mid s \subseteq x \right\}, \qquad s \in {}^{<\omega}A.$$

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The higher analogue of the classical setting, obtained by replacing ω with an uncountable cardinal κ.

Generalized descriptive set theory: preliminary notions

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Let λ, μ be cardinals, let μ be infinite, $\lambda \geq 2$. Consider the set ${}^{\mu}\lambda$, equipped with the *bounded topology* τ_b, generated by the sets

$$N_s({}^{\mu}\lambda) := \{ x \in {}^{\mu}\lambda \mid s \subseteq x \}, \qquad s \in {}^{<\mu}\lambda$$

(equivalently, $s : u \to \lambda$ for some bounded $u \subseteq \mu$).

- The weight of $({}^{\mu}\lambda, \tau_b)$ is: $\lambda^{<\mu} = |{}^{<\mu}\lambda| = \sup_{\nu<\mu} \lambda^{\nu}$.
- $({}^{\mu}\lambda, \tau_b)$ is completely metrizable if and only if $\operatorname{cof}(\mu) = \omega$.

The generalized Cantor space

- The classical Cantor space

$$^{\omega}2$$

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The weight of $({}^{\kappa}2, \tau_b)$ is $2^{<\kappa}$.

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The setup

We require that κ satisfies the condition $2^{<\kappa} = \kappa$.

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Lemma

Let κ be an infinite cardinal. Then $\kappa^{<\kappa} = \kappa$ is equivalent to κ *regular* and $2^{<\kappa} = \kappa$.

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If κ is singular?

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- $({}^{\kappa}\kappa, \tau_b)$ has weight $\kappa^{<\kappa} \geq \kappa^{\mathrm{cof}(\kappa)} > \kappa$.

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Proposition

If κ is a singular cardinal and $2^{<\kappa} = \kappa$, then:

- every non-empty $A \subseteq {}^{\kappa}2$ is a continuous image of ${}^{\kappa}\kappa$;
- every $A \subseteq {}^{\kappa}2$ is an injective continuous image of some closed $C \subseteq {}^{\kappa}\kappa$.

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Dimonte-Motto Ros (2021, in preparation)

If $\mathrm{cof}(\kappa) = \omega$, the generalized Baire Space is ${}^{\omega}\kappa$.

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  $${}^{\mathrm{cof}(\kappa)}\kappa$$

  The weight of $({}^{\mathrm{cof}(\kappa)}\kappa, \tau_b)$ is $\kappa^{<\mathrm{cof}(\kappa)}$.

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Lemma

If $2^{<\kappa} = \kappa$, then $\kappa^{<\mathrm{cof}(\kappa)} = \kappa$.

Classical definitions

Let $(X, \tau) = ({}^{\omega}2, \tau)$.

- Borel sets: ω_1-algebra generated by τ, i.e. the smallest collection of subsets of ${}^{\omega}2$ containing all open sets and closed under complements and unions of size $\leq \omega$.
- Analytic sets: continuous images of closed subsets of the Baire space ${}^{\omega}\omega$ (equivalently: it is either empty or the continuous image of the Baire space ${}^{\omega}\omega$).

Generalized definitions

Let $(X, \tau) = ({}^{\kappa}2, \tau_b)$.

- κ^+-Borel sets $\mathrm{Bor}(\kappa^+)$: κ^+-algebra generated by τ_b, i.e. the smallest collection of subsets of ${}^{\kappa}2$ containing all open sets and closed under complements and unions of size $\leq \kappa$.
- κ-Analytic sets $\mathbf{\Sigma}^1_1(\kappa)$: continuous images of closed subsets of the Baire space ${}^{\mathrm{cof}(\kappa)}\kappa$.

κ-Borel* sets

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Definition

A (generalized) tree $\mathscr{T}$ is a structure $(\mathscr{T}, \leq_{\mathscr{T}})$, where $\mathscr{T}$ is a set whose elements are called *nodes* and $\leq_{\mathscr{T}}$ is a nonempty partial order on $\mathscr{T}$ such that:

1. There is a unique element $r \in \mathscr{T}$, called *root*, such that $r \leq_{\mathscr{T}} p$ for all $p \in \mathscr{T}$.
2. For every $p \in \mathscr{T}$, the set $\mathrm{Pred}_{\mathscr{T}}(p) = \{p' \in \mathscr{T} \mid p' <_{\mathscr{T}} p\}$ of all predecessors of p is well-ordered by $\leq_{\mathscr{T}}$.
3. Given $p, p' \in \mathscr{T}$ such that $\mathrm{Pred}(p) = \mathrm{Pred}(p')$ and the order type of $\mathrm{Pred}(p)$ is a limit ordinal, then $p = p'$.

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A *leaf* is a terminal node of $\mathscr{T}$, i.e. a node $p \in \mathscr{T}$ such that $p \nless_{\mathscr{T}} p'$ for every $p' \in \mathscr{T}$. We denote by $\partial\mathscr{T}$ the collection of all leaves of $\mathscr{T}$.
A *branch* in $\mathscr{T}$ is a maximal chain in $\mathscr{T}$.

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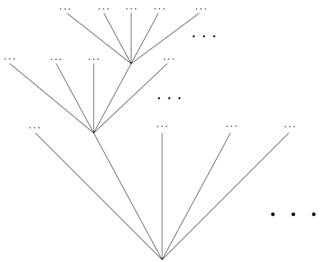
A tree is a $\kappa^+\text{cof}(\kappa)$-tree if all branches have length $< \text{cof}(\kappa)$ and all nodes have at most κ-many immediate successors.



κ-Borel* sets

Definition

A κ-Borel* code is a pair $(\mathscr{T}, \ell)$ such that $\mathscr{T}$ is a $\kappa^+\text{cof}(\kappa)$-tree in which every chain has a (unique) supremum and ℓ is the labelling function

$$\ell : \partial\mathscr{T} \longrightarrow \mathbf{\Delta}^0_1(\kappa^+)$$

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Given $\xi \in {}^\kappa 2$, the κ-Borel* game on $(\mathcal{T}, \ell)$ is $G(\mathcal{T}, \ell, \xi)$. There are two players, I and II, taking turns on the tree $\mathcal{T}$. The game ends when the players have selected a leaf $b \in \partial\mathcal{T}$: Player II wins if and only if $\xi \in \ell(b)$, otherwise I wins. The set coded by $(\mathcal{T}, \ell)$ is:

$$B(\mathcal{T}, \ell) = \{\xi \in {}^\kappa 2 \mid \mathrm{II} \uparrow G(\mathcal{T}, \ell, \xi)\}.$$

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Definition

A set $A \subseteq {}^\kappa 2$ is a κ-Borel* set if it admits a κ-Borel* code, that is, if there is a κ-Borel* code $(\mathscr{T},\ell)$ such that $B(\mathscr{T},\ell) = A$.

Overview

	$\kappa = \omega$	$\kappa > \omega$ singular and $\mathrm{cof}(\kappa) = \omega$	$\kappa > \omega$ singular and $\mathrm{cof}(\kappa) > \omega$	$\kappa > \omega$ regular
${}^{\kappa}2 \approx {}^{\mathrm{cof}(\kappa)}\kappa$	No	Yes	Yes	Yes, iff κ is not weakly compact
Borel hierarchy	Unique, length ω_1	Double, length κ^+		Unique, length κ^+
$\mathbf{\Sigma}^1_1(\kappa) =$ cont. images of ${}^{\mathrm{cof}(\kappa)}\kappa$	Yes	Yes	No	No
$\mathrm{Bor}(\kappa^+)$ vs. $\mathbf{\Sigma}^1_1(\kappa)$	$\mathrm{Bor} = \mathbf{\Delta}^1_1 \subsetneq \mathbf{\Sigma}^1_1$		$\mathrm{Bor} \subsetneq \mathbf{\Delta}^1_1 \subsetneq \mathbf{\Sigma}^1_1$	
$\mathrm{Bor}^*(\kappa)$	$\mathrm{Bor} = \mathrm{Bor}^* = \mathbf{\Delta}^1_1$		$\mathbf{\Delta}^1_1 \subseteq \mathrm{Bor}^* \subseteq \mathbf{\Sigma}^1_1$	
Unfair κ-Borel* codes	No	No	Yes	Yes