Distributivity spectra and fresh functions

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Winter School in Abstract Analysis 2022
1st of February 2022

Definition

$\mathbb{P}$ is λ-distributive if it does not add a function $f : \lambda \to Ord$ with $f \notin V$.
$\mathfrak{h}(\mathbb{P}) :=$ least λ such that $\mathbb{P}$ is not λ-distributive (the distributivity of $\mathbb{P}$).

Let $\lambda = \mathfrak{h}(\mathbb{P})$, and let $f : \lambda \to Ord$ witness this, i.e., $f \notin V$.
Note that $f \restriction \gamma \in V$ for every $\gamma < \lambda$ (f is not just new, but even "fresh").

Definition (Fresh function spectrum)

We say that $\lambda \in \mathrm{FRESH}(\mathbb{P})$ if in some extension of V by $\mathbb{P}$,

there exists a fresh function on λ,

i.e., a function $f : \lambda \to Ord$ with

1. $f \notin V$, but
2. $f \restriction \gamma \in V$ for every $\gamma < \lambda$.

Note: $\lambda \in \mathrm{FRESH}(\mathbb{P}) \iff \mathrm{cf}(\lambda) \in \mathrm{FRESH}(\mathbb{P})$

So from now on, we only talk about regular cardinals λ.

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- $\min(\mathrm{FRESH}(\mathbb{P})) = \mathfrak{h}(\mathbb{P})$

Proposition

If $\lambda > |\mathbb{P}|$, then $\lambda \notin \mathrm{FRESH}(\mathbb{P})$.

Proof (Sketch).

- assume towards contradiction that $\dot{f} : \lambda \to Ord$ is fresh
- for each $\gamma < \lambda$, fix $p_\gamma \in \mathbb{P}$ such that p_γ decides $\dot{f} \restriction \gamma$
- λ regular, so there exists p^* with $p_\gamma = p^*$ for unboundedly many γ
- so p^* decides $\dot{f}$ (and hence $\dot{f}$ is not new) □

- $\mathrm{FRESH}(\mathbb{P}) \subseteq [\mathfrak{h}(\mathbb{P}), |\mathbb{P}|]$

Example: Let $\mathbb{C}$ be the usual ω-Cohen forcing (with $|\mathbb{C}| = \omega$).

$$\mathrm{FRESH}(\mathbb{C}) = \{\omega\}$$

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Theorem

If $\mathbb{P}$ satisfies $\mathbb{P} \times \mathbb{P}$ is χ-c.c. and $\lambda \geq \chi$, then $\lambda \notin \text{FRESH}(\mathbb{P})$.

Let $\mathbb{C}_\mu$ be the forcing for adding μ many ω-Cohen reals (μ arbitrary).

$$\text{FRESH}(\mathbb{C}_\mu) = \{\omega\}$$

Is $\mathbb{P}$ being χ-c.c. sufficient? No: consider a Suslin tree T (on ω_1)

Theorem

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Lemma (Kurepa)

Let λ be a regular cardinal and $\chi < \lambda$, and let T be a tree of height λ all whose levels are of size less than χ. Then

- T has a cofinal branch;
- (assuming T is well-pruned) there exists $\gamma < \lambda$ such that T does not split above level γ.

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Possible fresh function spectra under GCH

Let A be a set of regular cardinals.

Is there a (homogeneous!) forcing $\mathbb{P}$ *such that* $FRESH(\mathbb{P}) = A$*?*

Examples under GCH:

- $\mathrm{FRESH}(\mathbb{C}(\lambda)) = \{\lambda\}$
  - $<\lambda$-closed
  - $|\mathbb{C}(\lambda)| = \lambda$
- $\mathrm{FRESH}(\mathbb{C} \times \mathbb{C}(\omega_1)) = \{\omega, \omega_1\}$
  - If a function is fresh, it remains fresh in any extension.
- $\mathrm{FRESH}(\mathbb{C}(\omega_1) \times \mathbb{C}(\omega_3)) = \{\omega_1, \omega_3\}$
- More generally: if A is finite, then $\mathrm{FRESH}(\prod_{\lambda \in A} \mathbb{C}(\lambda)) = A$

What if A is infinite?

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A set A of regular cardinals is Easton closed if for every limit point α of A,

- α regular, not Mahlo $\Rightarrow \alpha \in A$,
- α singular $\Rightarrow \alpha^+ \in A$.

The Easton-closure (A) is … (what you would guess).

Let ${}^E\prod_{\lambda \in A} \mathbb{P}_\lambda$ denote the Easton product of the $\mathbb{P}_\lambda$:

- full support at singular limits,
- bounded support at regular limits (i.e., inaccessibles).

Theorem (GCH)

Let A be a set of regular cardinals. Then

$$\mathrm{FRESH}({}^E\textstyle\prod_{\lambda \in A} \mathbb{C}(\lambda)) = \text{Easton-closure}(A).$$

- If A is Easton closed, then $\mathrm{FRESH}({}^E\prod_{\lambda \in A} \mathbb{C}(\lambda)) = A$.
- So each Easton closed set appears as a fresh function spectrum!

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Why Easton closure?

Example

Let α be the first inaccessible and $A \subseteq \alpha$ an unbounded subset of regular cardinals. Then ${}^{E}\prod_{\lambda \in A} \mathbb{C}(\lambda)$ adds an α-Cohen real.

${}^{E}\prod_{\lambda \in A} \mathbb{C}(\lambda)$ is bounded support product and hence adds an α-Cohen real.

In particular, $\alpha \in \mathit{FRESH}({}^{E}\prod_{\lambda \in A} \mathbb{C}(\lambda))$.

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Let α be a Mahlo cardinal and $A \subseteq \alpha$. Then $\alpha \notin \mathrm{FRESH}({}^{E}\prod_{\lambda \in A} \mathbb{C}(\lambda))$.

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Example (GCH)

Let $A \subseteq \aleph_\omega$ be an unbounded subset of regular cardinals. Then ${}^{E}\prod_{\lambda \in A} \mathbb{C}(\lambda)$ adds an $\aleph_\omega^+$-Cohen real.

Follows from a paper of Shelah, which uses pcf theory.

In particular, $\aleph_\omega^+ \in$ *FRESH*$({}^{E}\prod_{\lambda \in A} \mathbb{C}(\lambda))$.

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In particular, $\aleph_\omega^+ \in FRESH({}^E\prod_{\lambda \in A} \mathbb{C}(\lambda))$.

- Easton closed sets are realizable by Easton products of Cohen forcings.
- Such products do not collapse any cardinals.

Conjecture (GCH?)

Whenever $\mathbb{P}$ does not collapse cardinals, then FRESH($\mathbb{P}$) is Easton closed.

If we allow $\mathbb{P}$ to collapse cardinals, we can do a bit more.

Proposition

If $\mathbb{P}$ collapses λ to $\mathfrak{h}(\mathbb{P})$, then $\lambda \in \text{FRESH}(\mathbb{P})$.

It follows that $[\mathfrak{h}(\mathbb{P}), \lambda] \subseteq$ *FRESH*$(\mathbb{P})$.

Example (under CH): Let $Coll(\omega_1, \omega_2)$ be the forcing collapsing ω_2 to ω_1 with countable conditions.

$$\text{FRESH}(Coll(\omega_1, \omega_2)) = \{\omega_1, \omega_2\}$$

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If we allow $\mathbb{P}$ to collapse cardinals, we can do a bit more.

Proposition

If $\mathbb{P}$ collapses λ to $\mathfrak{h}(\mathbb{P})$, then $\lambda \in \text{FRESH}(\mathbb{P})$.

It follows that $[\mathfrak{h}(\mathbb{P}), \lambda] \subseteq \mathit{FRESH}(\mathbb{P})$.

Example (under CH): Let $\mathit{Coll}(\omega_1, \omega_2)$ be the forcing collapsing ω_2 to ω_1 with countable conditions.

$$\text{FRESH}(\mathit{Coll}(\omega_1, \omega_2)) = \{\omega_1, \omega_2\}$$

- Easton closed sets are realizable by Easton products of Cohen forcings.
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- FRESH($Coll(\aleph_0, \aleph_\omega)$) $= \{\aleph_n : n \in \omega\}$
  - since $\aleph_{\omega+1}$ does not belong to it, this set is not Easton closed
- More generally: for ξ singular, and μ regular with $\mu \leq \mathrm{cf}(\xi)$, FRESH($Coll(\mu, \xi)$) $= [\mu, \xi)$
  - since ξ^+ does not belong to it, this set is not Easton closed
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We do not know whether non-trivial spectra below $\aleph_\omega$ are possible:

Question

Let $A \subsetneq \{\aleph_n : n \in \omega\}$ be infinite. Is $\text{FRESH}(\mathbb{P}) = A$ for some $\mathbb{P}$?

What about non-trivial spectra below a (non-Mahlo) inaccessible ξ?

Question

Let $A \subseteq \xi$ be an unbounded and co-unbounded set of regular cardinals.
Is $\text{FRESH}(\mathbb{P}) = A$ for some $\mathbb{P}$?

Omitting fresh function spectra

Let A be a set of regular cardinals.

Does there (always?) exist a forcing $\mathbb{P}$ such that FRESH($\mathbb{P}$) $= A$?

Remember:

$$\text{FRESH}(\mathbb{C}) = \{\omega\}$$

Is there a forcing $\mathbb{P}$ with FRESH($\mathbb{P}$) $= \{\omega_1\}$?

Yes, if:

- CH holds (then $\mathbb{C}(\omega_1)$ does the job),
- there exists a Suslin tree (then the Suslin tree does the job).

Is it provable in ZFC?

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Definition

Todorčević's maximality principle is the following assertion:

If $\mathbb{P}$ is a forcing which adds a fresh subset of ω_1, then

- $\mathbb{P}$ collapses ω_1, or
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It is consistent, relative to the existence of an inaccessible cardinal.

Theorem

Assume Todorčević's maximality principle, and "$0^\#$ does not exist".

Then for every forcing with $\omega_1 \in \text{FRESH}(\mathbb{P})$,

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Tree forcings on ω

- c.c.c. forcings:
  - Cohen forcing
  - random forcing
  - Hechler forcing

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- Axiom A forcings, with antichains of size continuum
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Recall that $\mathfrak{h} = \mathfrak{h}(\mathcal{P}(\omega)/\mathit{fin})$.

Theorem (Base Matrix Theorem; Balcar-Pelant-Simon)

$\mathcal{P}(\omega)/\text{fin}$ collapses $\mathfrak{c}$ to $\mathfrak{h}$.

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$\text{FRESH}(\mathcal{P}(\omega)/\text{fin}) = [\mathfrak{h}, \mathfrak{c}]$.

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- FRESH(*Mathias*) $= \{\omega\} \cup [\mathfrak{h}, \mathfrak{c}]$
  - Proof is based on *Mathias* $\cong \mathcal{P}(\omega)/\mathrm{fin} * $ *Mathias*(G)
- FRESH(*Sacks*) $= \{\omega\}$
  - Proof is somewhat similar to the proof of minimality
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Let κ be a measurable cardinal.

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Thank you for your attention and enjoy the Winter School...

Vienna, Stephansplatz (first district), during first lockdown, 9th April 2020

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Vienna, Graben (first district), during first lockdown, 9th April 2020

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Old KGRC (Josephinum), during first lockdown, 9th April 2020

Thank you for your attention and enjoy the Winter School...

Augarten, 3rd December 2020