Aronszajn trees and Kurepa trees

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1. *A λ-Aronszajn tree is a λ-tree which has no branch.*
2. *A λ^+-tree T is called special if there exists a function $f: T \to \lambda$ such that for $s, t \in T$, if $s <_T t$, then $f(s) \neq f(t)$.*

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2. *A λ^+-tree T is called special if there exists a function $f: T \to \lambda$ such that for $s, t \in T$, if $s <_T t$, then $f(s) \neq f(t)$.*
3. *A λ-Kurepa tree is a λ-tree which has more than λ many branches.*

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  - Therefore, a Suslin tree cannot be special
  - ...you can also view it in the following way: if there were a special Suslin tree, forcing with it would result in a special tree which has a branch, so λ^+ would be collapsed, contradicting the λ^+-c.c. of the Suslin tree.

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1. A λ-*Aronszajn tree* is a λ-tree which has no branch.
2. A λ^+-tree T is called *special* if there exists a function $f : T \to \lambda$ such that for $s, t \in T$, if $s <_T t$, then $f(s) \neq f(t)$.

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1. *There are no $\aleph_0$-Aronszajn trees. (König's Lemma)*
2. *There are always $\aleph_1$-Aronszajn trees.*
3. *the existence of $\aleph_2$-Aronszajn trees is independent of ZFC:*
    - *under CH, there exists an $\aleph_2$-Aronszajn tree*
    - *in Mitchell's model, the tree property on $\aleph_2$ holds (needs a weakly compact cardinal)*

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4. *If $2^\lambda = \lambda^+$, then there exists a special λ^{++}-Aronszajn tree.*

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4. *If $2^\lambda = \lambda^+$, then there exists a special λ^{++}-Aronszajn tree.*
5. *If κ is inaccessible, then*

   there is a κ-Aronszajn tree $\iff$ κ is not weakly compact.

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2. *the existence of $\aleph_1$-Kurepa trees is independent of ZFC:*
    - *under $\diamondsuit^+$ (i.e., in particular in $V = L$), there exists an $\aleph_1$-Kurepa tree*
    - *it is consistent that there exists no $\aleph_1$-Kurepa tree (needs an inaccessible cardinal)*

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and there exists no $\aleph_n$*-Kurepa tree.*

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all $\aleph_2$-Aronszajn trees are special,

and there exists no $\aleph_2$-Kurepa tree and no $\aleph_1$-Kurepa tree.

Theorem (Laver-Shelah)

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Theorem (Baumgartner-Malitz-Reinhardt)

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There exists a model of ZFC in which

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Specializing $\aleph_1$-trees

Definition

Let T be an $\aleph_1$-Aronszajn tree. Let $\mathbb{S}(T)$ be the forcing consisting of conditions p satisfying the following:

1. $p\colon T \to \omega$ *is a finite partial function*
2. *if* $s, t \in \mathrm{dom}(p)$ *and* $s <_T t$, *then* $p(s) \neq p(t)$.

The order is given by $q \leq p$ *if* $q \supseteq p$.

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   - the generic function $f\colon T \to \omega$ is total,
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2. if T has a branch b, then $\mathbb{S}(T)$ adds an injection $f\colon b \to \omega$
    - ω_1 is collapsed to ω.
3. If T is Aronszajn, then $\mathbb{S}(T)$ does not collapse cardinals
    - it has the c.c.c. (difficult)

Theorem (Baumgartner-Malitz-Reinhardt)

There exists a model of ZFC in which

there exists an $\aleph_1$-Aronszajn tree, (always true)

all $\aleph_1$-Aronszajn trees are special.

1. Start with a model of $2^{\aleph_1} = \aleph_2$.

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6. Use a bookkeeping to specialize all $\aleph_1$-Aronszajn trees.

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4. $2^{\aleph_1} = \aleph_2$ stays true during the iteration.
5. Since $\aleph_1$-Aronszajn trees have size $\aleph_1$, there are only $\aleph_2$ many.
6. Use a bookkeeping to specialize all $\aleph_1$-Aronszajn trees.
7. If T is special, then it stays special.

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Back to $\aleph_2$-trees:

Theorem (Laver-Shelah)

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In fact, their aim was to get a model of

$$CH + \text{no } \aleph_2\text{-Suslin tree.}$$

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- $\mathbb{S}(T)$ has the c.c.c.

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- Want $\mathbb{S}(T)$ to have the $\aleph_2$-c.c. (needs a weakly compact)

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Lemma (Laver-Shelah)

If κ_2 is weakly compact, then in the extension by $\mathrm{col}(\aleph_1, {<}\kappa_2)$,

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1. Use a countable support iteration of length ω_3 and bookkeeping to specialize all $\aleph_2$-Aronszajn trees.

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1. Use a countable support iteration of length ω_3 and bookkeeping to specialize all $\aleph_2$-Aronszajn trees.
2. It is shown "by hand" that also the iteration has the $\aleph_2$-c.c.

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1. Use a countable support iteration of length ω_3 and bookkeeping to specialize all $\aleph_2$-Aronszajn trees.
2. It is shown "by hand" that also the iteration has the $\aleph_2$-c.c.
3. Warning: there is no iteration theorem for countable support iterations with $\aleph_2$-c.c. iterands.

Theorem (Laver-Shelah)

There exists a model of ZFC in which

- *there exists an $\aleph_2$-Aronszajn tree,*
- *all $\aleph_2$-Aronszajn trees are special.*

There exists an $\aleph_2$-Aronszajn tree, because:

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There exists an $\aleph_2$-Aronszajn tree, because:

1. $\mathrm{col}(\aleph_1, {<}\kappa_2)$ collapses $2^{\aleph_0}$ to $\aleph_1$.
2. Both $\mathrm{col}(\aleph_1, {<}\kappa_2)$ and the subsequent iteration are σ-closed, so there are no new reals added.

Theorem (Laver-Shelah)

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1. $\mathrm{col}(\aleph_1, {<}\kappa_2)$ collapses $2^{\aleph_0}$ to $\aleph_1$.
2. Both $\mathrm{col}(\aleph_1, {<}\kappa_2)$ and the subsequent iteration are σ-closed, so there are no new reals added.
3. So CH holds in the final model (which implies its existence).

Theorem

There exists a model of ZFC in which

there exists an $\aleph_2$-Aronszajn tree,

all $\aleph_2$-Aronszajn trees are special,

and there exists no $\aleph_2$-Kurepa tree and no $\aleph_1$-Kurepa tree.

How to avoid Kurepa trees?

Theorem (Silver)

Let λ be an inaccessible cardinal and $\mathbb{L} = \mathrm{col}(\aleph_n, <\lambda)$ with $n \geq 1$. There is no $\aleph_n$-Kurepa tree in the extension by $\mathbb{L}$.

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Let $\mathbb{R}$ be a forcing which is ${<}\aleph_n$-closed. Then $\mathbb{R}$ does not add branches to $\aleph_n$-trees.

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Lemma (Unger)

In V, let

- *$\mathbb{P}$ be a forcing which has the $\aleph_n$-c.c., and*
- *$\mathbb{R}$ be a forcing which is ${<}\aleph_n$-closed.*

In $V^{\mathbb{P}}$, let T be an $\aleph_n$-tree. Then forcing with $\mathbb{R}$ over $V^{\mathbb{P}}$ does not add branches to T.

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- Note that, in $V^{\mathbb{L}_{<\mu}*\mathbb{Q}}$, we have
  - $2^{\aleph_n} < \lambda$,
  - and hence $|[T]| < \lambda$.

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  2. If $\mathbb{Q}$ has the $\aleph_n$-c.c., Unger's Lemma implies that $\mathbb{L}_{[\mu,\lambda)}$ does not add branches to T.
- Thus T has less than $\lambda = \aleph_{n+1}$ many branches in $V^{\mathbb{L}*\mathbb{Q}}$, so T is not a Kurepa tree.

Theorem

There exists a model of ZFC in which

there exists an $\aleph_2$-Aronszajn tree,

all $\aleph_2$-Aronszajn trees are special,

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  - κ_2 is supercompact, and
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- Then use forcings $\mathbb{S}(T)$ to specialize all $\aleph_2$-Aronszajn trees.
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- CH holds in final model: so there exists an $\aleph_2$-Aronszajn tree.

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- It remains to show that in the final model $V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\omega_3}}$,
  1. there are no $\aleph_1$-Kurepa trees, and
  2. there are no $\aleph_2$-Kurepa trees.

No $\aleph_1$-Kurepa trees

Lemma (Capturing $\aleph_1$-trees by suitable subforcings)

Let T be an $\aleph_1$-tree in $V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\omega_3}}$. Then, in $V^{\mathbb{L}_2}$, there exists $\mathbb{Q}$ such that

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- Apply the generalization of Silver's Theorem: T is not an $\aleph_1$-Kurepa tree in $V^{\mathbb{L}_2 * \mathbb{Q}}$.

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No $\aleph_2$-Kurepa trees

We use the following lemma:

Lemma (Mitchell)

Let $\mathbb{P}$ be a forcing where $\mathbb{P} \times \mathbb{P}$ has the $\aleph_n$-c.c.. Then $\mathbb{P}$ does not add branches to trees of height $\aleph_n$.

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- Since $|\mathbb{S}_\alpha| \leq \aleph_2$, by the generalization of Silver's Theorem, T is not an $\aleph_2$-Kurepa tree in $V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha}$.

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Let $\mathbb{P}$ be a forcing where $\mathbb{P} \times \mathbb{P}$ has the $\aleph_n$-c.c.. Then $\mathbb{P}$ does not add branches to trees of height $\aleph_n$.

- Let $T \in V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\omega_3}}$ be an $\aleph_2$-tree.
- Since $\mathbb{S}_{\omega_3}$ has the $\aleph_2$-c.c., there exists $\alpha < \omega_3$ such that $T \in V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha}$.
- Since $|\mathbb{S}_\alpha| \leq \aleph_2$, by the generalization of Silver's Theorem, T is not an $\aleph_2$-Kurepa tree in $V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha}$.
- $\mathbb{S}_{\omega_3}/\mathbb{S}_\alpha$ has the $\aleph_2$-c.c., and even $\mathbb{S}_{\omega_3}/\mathbb{S}_\alpha \times \mathbb{S}_{\omega_3}/\mathbb{S}_\alpha$ has the $\aleph_2$-c.c., so $\mathbb{S}_{\omega_3}/\mathbb{S}_\alpha$ does not add branches to T.

No $\aleph_2$-Kurepa trees

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- So T is not an $\aleph_2$-Kurepa tree in $V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\omega_3}}$.

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- Capture an $\aleph_n$-tree with a subforcing of size at most $\aleph_n$.
- Show that it is not an $\aleph_n$-Kurepa tree here.
- Show that the quotient forcing does not add branches.

Theorem

It follows from a proper class of supercompact cardinals, that there exists a model of ZFC in which for all regular cardinals κ

there exists a κ^+-Aronszajn tree,

all κ^+-Aronszajn trees are special,

and there exists no κ^+-Kurepa tree.

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and there exists no κ^+-Kurepa tree.

- Use an Easton support iteration to combine the forcings which work for ω-many successive regular cardinals.

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Question

Can we specialize trees of height $\aleph_n$ which have no cofinal branches but levels of size $\geq \aleph_n$? Is it possible to specialize these trees and control the existence of $\aleph_n$-Kurepa trees at the same time?

We can also be more precise about the Kurepa trees:

Question

Can we control the exact number of branches of the $\aleph_n$-Kurepa trees in a model in which all $\aleph_n$-Aronszajn trees are special?

Question

Is it possible to specialize Aronszajn trees while keeping limit cardinals strong limit?

Question

Is it possible to obtain a model for the main theorem (at all successors of regulars) in which there are still inaccessibles?