Tree-representations for Borel functions

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Borel sets

Definition 1.

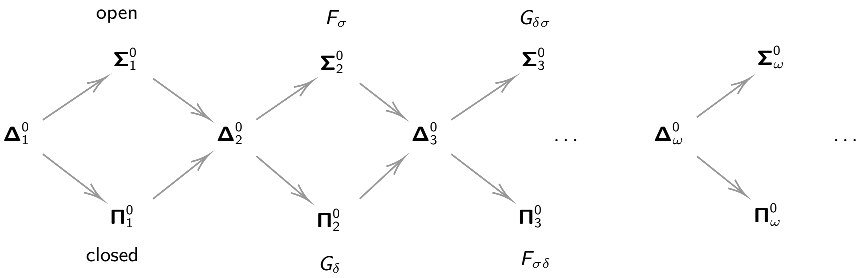
Let (X, τ) be a topological space. The class of *Borel sets* of X, denoted with $\mathcal{B}(X)$, is the σ-algebra generated by the open sets of X, i.e. the smallest σ-algebra containing the topology.

Definition 2.

Given two topological spaces X, Y and a function $f : X \to Y$, we say that f is a *Borel function* or *Borel measurable* if $f^{-1}(U) \in \mathcal{B}(X)$ for every open $U \subseteq Y$.

Borel Hierarchy

Take (X, τ) metrizable, we can stratify the Borel sets of X into classes $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi, \mathbf{\Delta}^0_\xi$ (for ξ countable ordinal) by inductively iterating countable unions and taking complements starting from the open sets.



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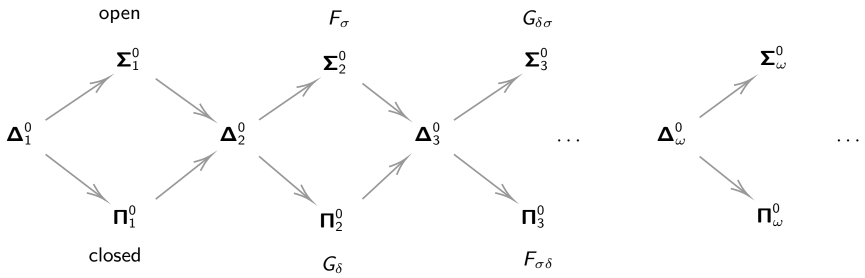

Definition 3.

Given two spaces X, Y, and a countable ordinal $\alpha > 1$, we say that a function $f : X \to Y$ is $\mathbf{\Sigma}^0_\alpha$-measurable if $f^{-1}(U) \in \mathbf{\Sigma}^0_\alpha(X)$ for every open $U \subseteq Y$.

Baire functions

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For all $\alpha > 1$ countable ordinals, we can define recursively the Baire class α to be the class of functions which are pointwise limits of sequences of Baire class $< \alpha$ functions.

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Theorem 6 (Lebesgue, Hausdorff, Banach).

Let X, Y be separable metrizable spaces, with X zero-dimensional. Then for $1 \leq \alpha < \omega_1$ $f : X \to Y$ is Baire class α if and only if it is $\mathbf{\Sigma}^0_{\alpha+1}$-measurable.

Trees

Definition 7.

A *Tree* on a set A is a subset $T \subseteq A^{<\omega} = \{\langle a_0, a_1, a_2, \ldots, a_{n-1}\rangle \mid n \in \omega \wedge \forall i < n \; a_i \in A\}$ closed under initial segments. The *body* of a tree T is the set if its *branches*:

$$[T] = \{(a_n)_{n\in\omega} \in A^\omega \mid \langle a_0, a_1, \ldots, a_n\rangle \in T \text{ for all } n \in \omega\}$$

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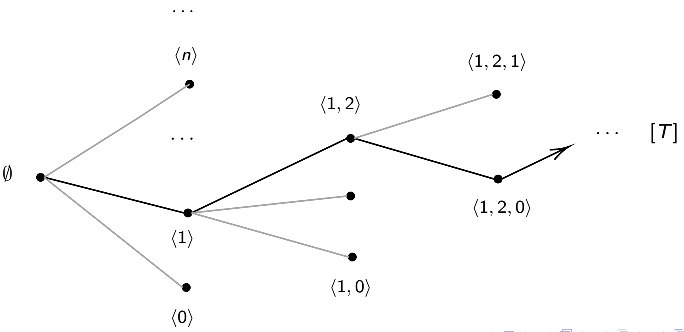

Topologies on Trees

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- the topology τ_C generated by the sets $\{T \text{ tree on } A \mid s \in T\}, \{T \text{ tree on } A \mid s \notin T\}$ with $s \in A^{<\omega}$.

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Remark 8.

1. $\tau_S \subseteq \tau_C$.
2. $\tau_C \subseteq \mathbf{\Sigma}^0_2(\tau_S)$.
3. $(\mathrm{Tr}(A), \tau_C) \cong 2^\omega$.
4. τ_S is the Scott topology of $(\mathrm{Tr}(A), \subseteq)$.

Game for Borel functions

Definition 9 (Borel Game).

Given a function $f : \omega^\omega \to \omega^\omega$ we define the following perfect information two players infinite game $G_{\mathrm{B}}(f)$:
At each round $n \in \omega$, Player I plays a natural number $x_n \in \omega$, and then Player II plays a finite tree T_n on $\omega \times \omega$ (i.e. the set of couples of natural numbers) s.t. $T_n \subseteq T_{n+1}$.

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$$(x_0, T_0, x_1, T_1, x_2, T_2, \ldots, x_n, T_n)$$

So at the end of the game Player I has produced an infinite sequence $x \in \omega^\omega$ whilst Player II has produced a tree $T = \bigcup_{n \in \omega} T_n$

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So at the end of the game Player I has produced an infinite sequence $x \in \omega^\omega$ whilst Player II has produced a tree $T = \bigcup_{n \in \omega} T_n$

We say that Player II wins iff T has a unique branch and Proj(branch of T) $= f(x)$.

$f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$

$I :$

$II :$

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$I:$ 3

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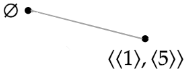
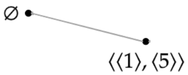


$f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$

$I :$ 3 10

$II :$



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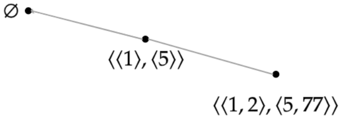

$f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$

$I :$ 3 10 4

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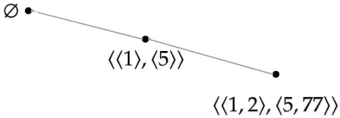

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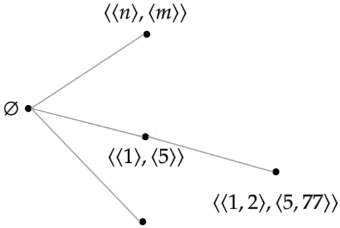

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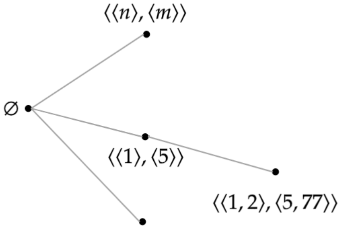

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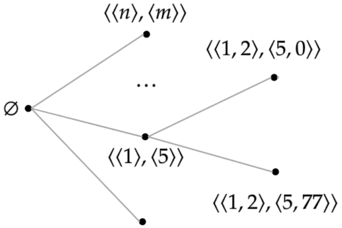
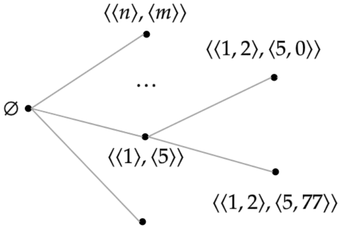


$f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$

$I:$ 3 10 4 56 0

$II:$



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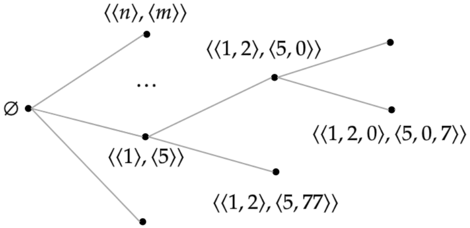

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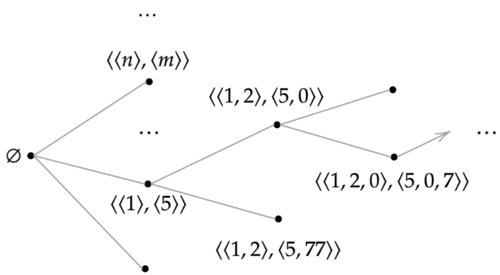

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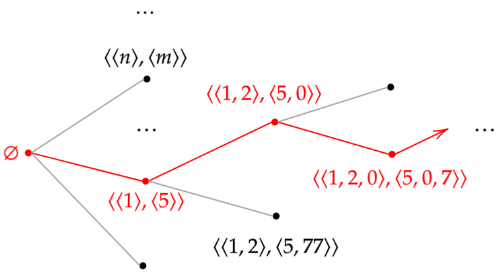
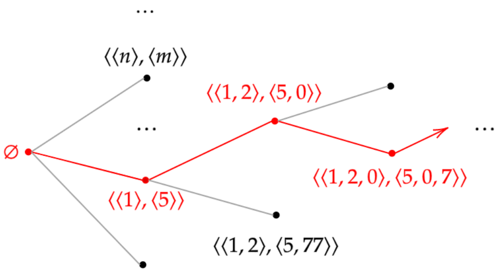


$f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$

$I:$ 3 10 4 56 0 $\ldots$ $\mapsto$ $x = \langle 3, 10, 4, \ldots \rangle$

$II:$



$\mapsto$ $\langle 1, 2, 0, \ldots \rangle \stackrel{?}{=} f(x)$

Strategies for Player II in $G_{\mathbf{B}}(f)$

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Given a strategy σ for Player II in $G_{\mathbf{B}}(f)$ then the following map is continuous

$$\begin{aligned} \varphi_\sigma : \omega^\omega &\longrightarrow (\mathrm{Tr}(\omega \times \omega), \tau_S) \\ x &\longmapsto \bigcup_{n \in \omega} \sigma(x \restriction n) \end{aligned}$$

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$$\varphi_\sigma : \omega^\omega \longrightarrow (\mathrm{Tr}(\omega \times \omega), \tau_S)$$
$$x \longmapsto \bigcup_{n\in\omega} \sigma(x \upharpoonright n)$$

Conversely, given a continuous function $\varphi : \omega^\omega \to (\mathrm{Tr}(\omega \times \omega), \tau_S)$, there exists a strategy σ_φ for Player II such that

$$\bigcup_{n\in\omega} \sigma_\varphi(x \upharpoonright n) = \varphi(x) \quad \text{for all } x \in \omega^\omega$$

Borel Representation result

Theorem 10 ([Semmes, 2009]).

A function $f : \omega^\omega \to \omega^\omega$ is Borel measurable if and only if Player II has a winning strategy in $G_{\mathbf{B}}(f)$.

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Theorem 11 (Louveau, 2009).

A function $f : \omega^\omega \to \omega^\omega$ is Borel measurable if and only if there exists a continuous function $\varphi : \omega^\omega \to (Tr(\omega \times \omega), \boldsymbol{\tau_C})$ such that, for all $x \in \omega^\omega$, $\varphi(x)$ has a unique branch and $Proj(\text{branch of } \varphi(x)) = f(x)$.

The map φ of Theorem 11 is called a *tree-representation* for the function f, and a function admitting such map is called *tree-representable*.

Proof(s) of Louveau's theorem

Proof of Louveau's theorem

$(\Leftarrow)$: Given a function $f : \omega^\omega \to \omega^\omega$ with a tree-representation $\varphi : \omega^\omega \to (\mathrm{Tr}(\omega \times \omega), \tau_C)$, and an open set $U \subseteq \omega^\omega$ we have

$$\begin{aligned} f^{-1}(U) &= \{x \in \omega^\omega \mid \exists y, z \in \omega^\omega \ (y \in U \wedge \forall n \in \omega \ \langle y \restriction n, z \restriction n \rangle \in \varphi(x))\} \\ &= \{x \in \omega^\omega \mid \forall y, z \in \omega^\omega \ (y \in U \vee \exists n \in \omega \ \langle y \restriction n, z \restriction n \rangle \notin \varphi(x))\} \end{aligned}$$

Hence $f^{-1}(U) \in \mathbf{\Delta}^1_1(\omega^\omega)$, and by Lusin's separation theorem it is Borel.

Proof of Louveau's theorem

$(\Rightarrow)$: Given a Borel function $f : \omega^\omega \to \omega^\omega$, there is a zero-dimensional Polish topology τ' on ω^ω which refines the usual product topology τ and such that $f \circ id : (\omega^\omega, \tau') \to (\omega^\omega, \tau)$ is continuous, with $id : (\omega^\omega, \tau') \to (\omega^\omega, \tau)$ being the identity.

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$$\begin{aligned} h : \omega^\omega &\longrightarrow \omega^\omega \times \omega^\omega \\ x &\longmapsto (f(x), g \circ id^{-1}(x)) \end{aligned}$$

The graph of h is closed as

$$\text{graph}(h) = \{(x, y, z) \in (\omega^\omega)^3 \mid y = f \circ id \circ g^{-1}(z),\ x = id \circ g^{-1}(z)\}$$

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therefore there is a pruned tree T on ω^3 such that $\text{graph}(h) = [T]$. Now we can set

$$\begin{aligned} \varphi : \omega^\omega &\longrightarrow (\text{Tr}(\omega \times \omega), \tau_C) \\ x &\longmapsto \{s \in (\omega \times \omega)^n \mid n \in \omega \text{ and } \langle x \restriction n, s \rangle \in T\} \end{aligned}$$

And φ is the tree-representation we were looking for. □

Ideas for another proof.

$(\Rightarrow)$: We can prove this direction also by induction on the Baire hierarchy, by showing that the pointwise limit of a sequence of tree-representable functions is itself tree-representable.

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($\Rightarrow$) : We can prove this direction also by induction on the Baire hierarchy, by showing that the pointwise limit of a sequence of tree-representable functions is itself tree-representable.
Indeed as every continuous function is tree-representable by a map that ranges among *linear* trees, we would be done.

Finer results

Given a Borel function $f : \omega^\omega \to \omega^\omega$ we now know that it is tree-representable, but how "complicated" are the trees in the range of the tree-representation?

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Intuitive answer

The more complex the function f, the more complex the trees in the representation

Rank of a tree

Given a tree T that does not have infinite branches (we say that T is *well-founded*) then we can define recursively the usual *rank* : $T \to \mathrm{Ord}$ as follows:

$$\mathrm{rank}_T(s) = \begin{cases} \sup\{\mathrm{rank}_T(s^\frown a) + 1 \mid s^\frown a \in T\} & \text{if } s \text{ is not terminal} \\ 0 & \text{otherwise} \end{cases}$$

where we call a node $s \in T$ *terminal* in T if there is no a such that $s^\frown a \in T$.

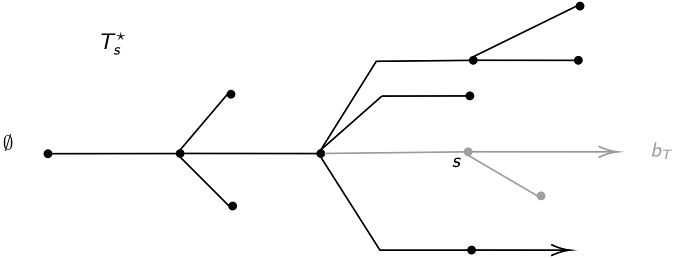
We can define the rank of a well-founded tree T as

$$\mathrm{rank}(T) = \mathrm{rank}_T(\emptyset) + 1$$

Rank* of a tree

Given a tree T and a node $s \in T$, define $T^\star_s = T \setminus (s^\frown(T \restriction s))$.
Suppose $T^\star_s$ is well-founded, then we set

$$\mathrm{rank}^\star_T(s) = \mathrm{rank}(T^\star_s).$$



Representing Baire class α functions

Stratifying UB

Using the rank_T and $\text{rank}^\star_T$ functions, we can define subclasses UB_α for each α countable ordinal, that stratify the class of trees having a unique branch

$$\text{UB}_0 \subset \text{UB}_1 \subset \cdots \subset \text{UB}_\alpha \subset \ldots$$

As we climb up the hierarchy we get trees that branch out more and more off the unique branch.

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As we climb up the hierarchy we get trees that branch out more and more off the unique branch.

Theorem 12 (Louveau, Semmes 2009).

For any $\alpha < \omega_1$, a function $f : \omega^\omega \to \omega^\omega$ is Baire class α if and only if there exists a continuous function $\varphi : \omega^\omega \to (Tr(\omega \times \omega), \tau_C)$ such that, for all $x \in \omega^\omega$, $\varphi(x)$ is in UB_α and $Proj(\text{branch of } \varphi(x)) = f(x)$.

Representating $\mathbf{\Sigma}^0_\lambda$-measurable functions

We can define new subclasses $\mathrm{UB}^l_\lambda \subset \mathrm{UB}_\lambda$ for each λ countable limit that allows to capture the class of $\mathbf{\Sigma}^0_\lambda$-measurable functions.

Theorem 13 (Louveau, 2009).

For any countable limit ordinal λ, a function $f : \omega^\omega \to \omega^\omega$ is $\mathbf{\Sigma}^0_\lambda$-measurable if and only if there exists a continuous function $\varphi : \omega^\omega \to (Tr(\omega \times \omega), \tau_C)$ such that, for all $x \in \omega^\omega$, $\varphi(x)$ is in UB^l_λ and $Proj(\text{branch of } \varphi(x)) = f(x)$.

Additional Representation results

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Given a function $f : \omega^\omega \to \omega^\omega$, we define the modified Borel game $G^w_{\mathbf{B}}(f)$ as the game in which Player I constructs a sequence $x \in \omega^\omega$ and Player II constructs a tree T on ω and Player II wins the game if T has a unique branch and its branch is $f(x)$.

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Proposition 15 (N.).

Given a function $f : \omega^\omega \to \omega^\omega$, Player II has a winning strategy in $G^w_{\mathbf{B}}(f)$ if and only if $graph(f) \in \mathbf{\Pi}^0_2$.

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Proposition 16 (N.).

Given a Borel function $f : \omega^\omega \to \omega^\omega$, if graph$(f) \notin \mathbf{\Pi}^0_2$ then Player I has a winning strategy in $G^w_{\mathbf{B}}(f)$.

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- Fix a decreasing sequence of open sets $(U_n)_{n\in\omega}$ s.t. $\text{graph}(f) = \bigcap_{n\in\omega} U_n$.
- Consider the strategy, which we call σ, for Player II according to which, at round $i \in \omega$, if Player I has played $s \in \omega^i$, Player II adds to his tree the sequences $t \in \omega^{<\omega}$ s.t. $\max\{n \in \omega \mid N_s \times N_t \subseteq U_n\} > \max\{n \in \omega \mid N_s \times N_{t\upharpoonright|t|-1} \subseteq U_n\}$.

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- Check that

$$y \in \left[\bigcup_{n\in\omega} \sigma(x \restriction n)\right] \iff \forall n \exists m_0 \exists m_1 \left(N_{x\restriction m_0} \times N_{y\restriction m_1} \subseteq U_n\right)$$

$$\iff \langle x, y\rangle \in \mathrm{graph}(f)$$

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($\Leftarrow$)

- Fix a decreasing sequence of open sets $(U_n)_{n\in\omega}$ s.t. $\mathrm{graph}(f) = \bigcap_{n\in\omega} U_n$.
- Consider the strategy, which we call σ, for Player II according to which, at round $i \in \omega$, if Player I has played $s \in \omega^i$, Player II adds to his tree the sequences $t \in \omega^{<\omega}$ s.t. $\max\{n \in \omega \mid N_s \times N_t \subseteq U_n\} > \max\{n \in \omega \mid N_s \times N_{t\restriction|t|-1} \subseteq U_n\}$.
- Check that

$$y \in \left[\bigcup_{n\in\omega} \sigma(x \restriction n)\right] \iff \forall n \exists m_0 \exists m_1 \, (N_{x\restriction m_0} \times N_{y\restriction m_1} \subseteq U_n)$$

$$\iff \langle x, y\rangle \in \mathrm{graph}(f)$$

($\Rightarrow$): Fix a winning strategy σ for Player II in $G^w_{\mathbf{B}}(f)$, check that

$$\mathrm{graph}(f) = \bigcap_{n\in\omega} \bigcup \{N_s \times N_t \mid t \in \omega^n \text{ and } s \in \omega^{<\omega} \text{ s.t. } t \in \sigma(s)\}$$

Additional Representation results

If we modify accordingly the Louveau's definition of tree-representable function with end up characterizing closed graph functions.

Proposition 17 (N.).

Given a function $f : \omega^\omega \to \omega^\omega$, *its graph is closed if and only if there exists a continuous function* $\varphi : \omega^\omega \to (Tr(\omega), \tau_C)$ *such that, for all* $x \in \omega^\omega$, $\varphi(x)$ *has a unique branch and its branch is* $f(x)$.

Other reduction games

From the Borel game $G_{\mathsf{B}}(f)$ we can recover other similar games (reduction games) that have been studied, by adding contraints on the complexity of the trees played by Player II.

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Definition 18.

Given G, G' perfect information two players infinite games, we say that G, G' are *equivalent* if given a winning strategy for Player I (resp. II) in G we can define a winning strategy for Player I (resp. II) in G' and vice versa.

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Proposition 19 (Folklore).

The Wadge game $G_W(f)$, Duparc's eraser game $G_e(f)$ and Van Wesep's backtrack game $G_{bt}(f)$ are equivalent to the Borel game $G_{\mathbf{B}}(f)$ once we require Player II to play, in order to win, a tree respectively linear, in UB_1 and in a subclass of UB_1.

	$G_{\mathbf{B}}(f)$ where Player II plays a tree in
$G_W(f)$	UB_0
$G_e(f)$	UB_1
$G_{bt}(f)$	UB_1^-

Determinacy

Definition 20.

A two player perfect information infinite game is *determined* if any of the two players has a winning strategy.

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Theorem 21 ([Carroy, 2014]).

For all functions $f : \omega^\omega \to \omega^\omega$, the Wadge $G_{\mathrm{W}}(f)$, the eraser game $G_e(f)$ and the backtrack game $G_{\mathrm{bt}}(f)$ are determined.

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The proof of this result does not appeal to Martin's Borel determinacy.

Is the Borel Game $G_{\mathbf{B}}(f)$ determined?

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Theorem 22 (N.).

Given a subset $A \subseteq \omega^\omega$, if Player I has a winning strategy in $G_{\mathbf{B}}(\mathbb{1}_A)$ then A contains a perfect set.

where the function $\mathbb{1}_A$ is the function

$$\mathbb{1}_A : \omega^\omega \longrightarrow \omega^\omega$$
$$x \longmapsto \begin{cases} \langle 1 \rangle^\omega & \text{if } x \in A \\ \langle 0 \rangle^\omega & \text{otherwise} \end{cases}$$

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Corollary 23.

The determinacy of $G_{\mathbf{B}}(f)$ for all $f : \omega^\omega \to \omega^\omega$ implies that every non-Borel subset of the Baire space has the perfect set property.

Corollary 24.

(ZFC) *There exists a function $f : \omega^\omega \to \omega^\omega$ such that $G_{\mathbf{B}}(f)$ is undetermined.*

On Borel reducibility

Definition 25.

For $A, B \subseteq \omega^\omega$, the game $G_{\mathbf{B}}(A, B)$ is a game with the same rules as the Borel game, but Player II wins if and only if

$$x \in A \iff \text{Proj(unique branch of } T) \in B$$

where x is the sequence played by Player I and T is the tree played by Player II.

On Borel reducibility

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Remark.

Given $A, B \subseteq \omega^\omega$, Player II has a winning strategy in $G_{\mathbf{B}}(A, B)$ if and only if $A \leq_{\mathbf{B}} B$, i.e. there exists a Borel function $f : \omega^\omega \to \omega^\omega$ such that $f^{-1}(B) = A$.

On Borel reducibility

$\mathsf{AD}^{\mathbf{B}}$

We denote with $\mathsf{AD}^{\mathbf{B}}$ the statement "For all $A, B \subseteq \omega^\omega$, the game $G_{\mathbf{B}}(A, B)$ is determined".

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We denote with $\mathsf{AD}^{\mathbf{B}}$ the statement "For all $A, B \subseteq \omega^{\omega}$, the game $G_{\mathbf{B}}(A, B)$ is determined".

$\mathsf{AD}^{\mathbf{B}}$ implies the following statement:

$$\text{for all } A, B \subseteq \omega^{\omega} \; A \leq_{\mathbf{B}} B \text{ or } B \leq_{\mathbf{B}} \neg A$$

which is called $\mathsf{SLO}^{\mathbf{B}}$ and is sufficient (in $(\mathsf{ZF} + \mathsf{DC}(\omega^{\omega}) + \mathsf{BP})$) to prove that $\leq_B$ is well-founded and the structure of its equivalence classes is isomorphic to the one for the Wadge (continuous) reduction (see [Andretta and Martin, 2003]).

Conclusion

Open question.

What is the consistency strength of "$\mathrm{Det}(G_{\mathbf{B}}(f))$ for all $f : \omega^\omega \to \omega^\omega$"? What is the relationship of such statement with other known determinacy statements?

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What is the consistency strength of $\mathsf{AD}^{\mathbf{B}}$?

Open question (see [Andretta, 2006]).

$(\mathsf{ZF} + \mathsf{DC}(\omega^\omega) + \mathsf{BP})$ Does $\mathsf{SLO}^{\mathbf{B}} \iff \mathsf{AD}^{\mathbf{B}} \iff \mathsf{SLO}^{\mathsf{W}}$ hold?

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Thank you for the attention

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