The Wadge Hierarchy on 0-dimensional Polish space

Joint work with R. Carroy and L. Motto Ros

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Definition

Let Z be a topological space. Given $A, B \subseteq Z$ we say that A Wadge reduces to B (briefly $A \leqslant_{\mathrm{W}} B$) if there exists a continuous $f : Z \to Z$ such that $f^{-1}(B) = A$.

Our class of spaces: zero-dimensional Polish spaces (P0D), that is zero-dimensional, separable and completely metrizable spaces.

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$\omega^\omega, 2^\omega, \mathbb{R}\setminus\mathbb{Q}, \alpha \in \omega_1$

Examples Not P0D

$\mathbb{R},\ \ell^p,\ \mathbb{Q}$

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Wadge Game in ω^ω: let $A, B \subseteq \omega^\omega$, $\mathrm{G}(A, B)$

I	a_1	a_2	a_3	$\dots$
II	b_1	p	b_2	$\dots$

(Wadge) $A \leqslant_{\mathrm{W}} B \Leftrightarrow \mathrm{II}$ has a winning strategy in $\mathrm{G}(A, B)$

Wadge Lemma

(AD) Let $A, B \subseteq \omega^\omega$ then $A \leqslant_{\mathrm{W}} B$ or $\omega^\omega \setminus B \leqslant_{\mathrm{W}} A$.

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I	a_1	a_2	a_3	...
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where p is for "pass", i.e. II can skip his turn.

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I	a_1	a_2	a_3	$\dots$	$\rightarrow a = (a_1, a_2, \dots) \in \omega^\omega$
II	b_1	p	b_2	$\dots$	$\rightarrow b = (b_1, b_2, \dots) \in \omega^\omega$

Winning condition for II:

$a \in A$ if and only if $b \in B$.

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Theorem (Martin-Monk)

(AD) $(\mathcal{P}(\omega^\omega), \leqslant_{\mathrm{W}})$ is a well-founded quasi-order, called the Wadge quasi-order.

Definition

A subset $A \subseteq \omega^\omega$ is **selfdual** if $A \leqslant_{\mathrm{W}} \omega^\omega \setminus A$, otherwise A is **non selfdual**. Moreover, if we consider (the Wadge class) $[A] = \{B \in \mathcal{P}(\omega^\omega) \mid A \leqslant_{\mathrm{W}} B, B \leqslant_{\mathrm{W}} A\}$, $[A]$ is selfdual if so is A.

Consider the **coarse class**, $[A] \cup [\omega^\omega \setminus A]$, the relation on them induced by the Wadge reduction is a well-order. We define the rank of a coarse class (Wadge class), an ordinal that uniquely identifies the coarse class.

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Definition

Let α be the rank of a coarse class, we say

- $\alpha \in \mathrm{SD}_{\omega^\omega}$, if the αth coarse class coming from a selfdual subset A;
- $\alpha \in \mathrm{NSD}_{\omega^\omega}$, if the αth coarse class coming from a non selfdual subset A.

This was due to Wadge (probably), although not stated like this.

Alternating duality: If $\alpha < \Theta_{\omega^\omega}$, $\alpha \in \mathrm{SD}_{\omega^\omega}$ if and only if $\alpha + 1 \in \mathrm{NSD}_{\omega^\omega}$.

Goal: describe the partial-order of the Wadge classes on zero-dimensional Polish spaces up to isomorphism!

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On ω^ω or 2^ω

1) Well-foundedness
2) SLO_W
3) Alternating duality
4) Study of limit with countable cofinality
5) Study of limit with uncountable cofinality
6) Length of the hierarchy

General P0D Z

1) ✓
2) ✓
3) ?
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Theorem (R. Carroy, L. Motto Ros, S.)

Let Z be zero-dimensional Polish space and let $\alpha < \Theta_Z$ be in NSD_Z. Then $\alpha + 1 \in \mathrm{SD}_Z$.

Remark

The proof of the theorem is different depending on whether Z is countable or not. If Z is uncountable this can be obtained as consequence of the description of the Wadge Hierarchy on the Baire space via tools in Carroy-Medini-Müller(2022), while in countable case one needs a constructive proof.

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Theorem (Wadge, Carroy-Medini-Müller)

Let Z be a zero-dimensional Polish space and $A \subseteq Z$. The set A is selfdual if and only if there exists a pairwise disjoint partition $(U_n)_{n\in\omega}$ of Z in clopen subsets such that for each $n \in \omega$ there exists $A_n \subseteq Z$ non selfdual subset satisfying

$$A_n <_{\mathrm{W}} A \text{ and } \bigcup_{n\in\omega} A_n \cap U_n = A.$$

Corollary

Let Z be a zero-dimensional Polish space and $\alpha < \Theta_Z$ a limit ordinal of countable cofinality. If $\alpha \in \mathrm{SD}_Z$ then Z is not compact.

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Proposition

Let Z be a zero-dimensional Polish space, let α, β be two coarse classes. If $\alpha < \beta$ and $\alpha, \beta \in \mathrm{SD}_Z$ then there exists $\rho \in \mathrm{NSD}_Z$ such that $\alpha < \rho < \beta$.

Alternating duality Theorem (Wadge, Carroy-Motto Ros-S.)

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Proposition

Let Z be a zero-dimensional Polish space and $\alpha < \Theta_Z$. If α is a limit ordinal with $\mathrm{cof}(\alpha) = \omega_1$ then $\alpha \in \mathrm{NSD}_Z$.

Remark

The case $Z = 2^\omega, \omega^\omega$ are due to Wadge.

Definition

Let Z be a topological space then we define the perfect kernel of Z as

$$\ker_p(Z) = \{x \in Z \mid x \text{ is an accumulation points of } Z\}$$

where $x \in Z$ is an accumulation point if and only if each neighborhood U of x is uncountable.

Theorem (Wadge, Carroy-Motto Ros-S.)

Let Z be an uncountable zero-dimensional Polish space and let α be a limit ordinal with $\mathrm{cof}(\alpha) = \omega$. If $\ker_p(Z)$ is not compact then $\alpha \in \mathrm{SD}_Z$.

What if $\ker_p(Z)$ is compact?

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Recall that the Cantor-Bendixon derivatives on a topological space Z are defined as follows: $\mathrm{CB}_0(Z) = Z$,
$\mathrm{CB}_{\gamma+1}(Z) = \mathrm{CB}_\gamma(Z) \setminus \{x \in \mathrm{CB}_\gamma(Z) \mid x$ is isolated points in $\mathrm{CB}_\gamma(Z)\}$,
$\mathrm{CB}_\lambda(Z) = \bigcap_{\mu<\lambda} \mathrm{CB}_\mu(Z)$ when λ is a limit ordinal.

Definition

Let $\mathrm{Comp}(Z) = \min\{\gamma \in \omega_1 \mid \mathrm{CB}_\gamma(Z)$ is compact$\}$, where $\mathrm{CB}_\gamma(Z)$ is the Cantor-Bendixon derivatives.

Theorem (Wadge, Carroy-Motto Ros-S.)

Let Z be a zero-dimensional Polish space such that $\ker_p(Z)$ compact, and let $\alpha < \Theta_Z$. If α is a limit ordinal with $\mathrm{cof}(\alpha) = \omega$ then

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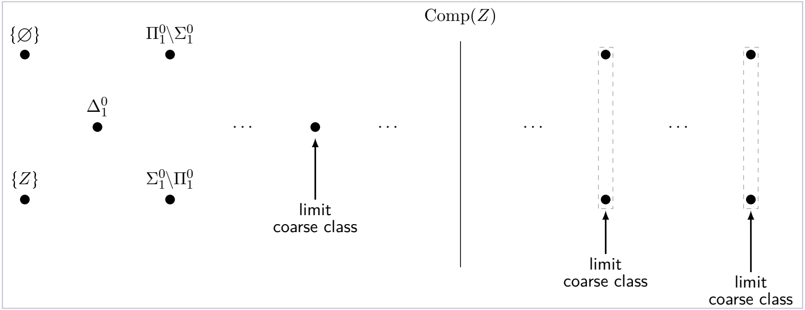
Comp(Z)
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$\Pi^0_1 \setminus \Sigma^0_1$
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Length of the Wadge hierarchy

Z is uncountable $\rightsquigarrow \Theta$.

Z is countable and $\mathrm{CB}(Z)$ is limit $\rightsquigarrow \mathrm{CB}(Z)$.

If Z is countable and $\mathrm{CB}(Z) = \lambda + n + 1$

	$\lvert \mathrm{CB}_{\lambda+n}(Z) \rvert = 1$	$\lvert \mathrm{CB}_{\lambda+n}(Z) \rvert > 1$
$\mathrm{Comp}\, Z < \lambda$	$\lambda + 2n + 2$	$\lambda + 2n + 3$
$\mathrm{Comp}\, Z > \lambda$	$\lambda + 2n + 1$	$\lambda + 2n + 2$

Thank you!