Hyperfinite graphings, part III

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Theorem (Bowen–Kun–S.)

Any bipartite hyperfinite a.e. one-ended regular graphing admits a **measurable perfect matching**.

Recall

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Structure of extreme points

If φ is an **extreme point** of C_G, then for a.e. edge $e \in E(G)$ we have

$$\varphi(e) \in \{0, \frac{1}{2}, 1\}$$

and the set of edges on which $\varphi = \frac{1}{2}$ is a **disjoint union of lines**, which we denote $L(\varphi)$.

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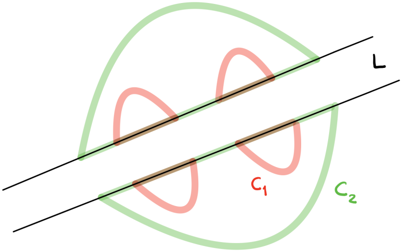
Connected toasts

Any hyperfinite a.e. one-ended regular graphing admits a **connected toast**.

Lemma

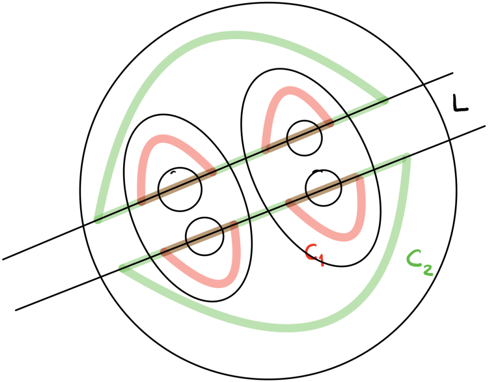
Suppose G is hyperfinite and one-ended and $L \subseteq G$ is a **family of disjoint lines of positive measure**. For every K there exists are Borel families $C_1, \ldots, C_K$, each consisting of **pariwise edge-disjoint cycles** such that

- each edge in $G \setminus L$ is **covered by at most one** of $C_1 \cup \ldots \cup C_K$,
- at least half of the edges in L are **covered by all** $C_1 \cap \ldots \cap C_K$



Proof

The proof uses a connected toast to inscribe cycles into bigger and bigger elements of the toast.



Claim

Any regular graphing admits a measurable fractional perfect matching τ which is positive on all its edges.

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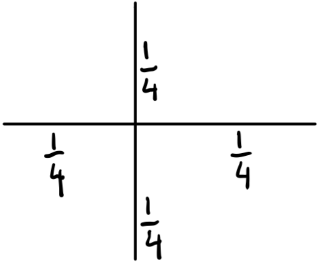
Proof

Put

$$\tau(e) = \frac{1}{d},$$

where d is the degree of G.



Lemma

Given an extreme point φ of C_G such that $\mu(L(\varphi)) > 0$ there **exists an extreme point** ψ of C_G such that

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Improvement measure

To **estimate** $\mu(L(\psi))$ for ψ in C_G we will use the fact that

$$\mu(L(\psi)) = 1 - 2\int_{E(G)} |\psi(e) - \frac{1}{2}| d\mu$$

Proof of the lemma

Given an extreme point φ of C_G such that $\mu(L(\varphi)) > 0$ we will find an **extreme point** ψ **of** C_G **such that**

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Proof

Choose K **very big** and λ **very small** and consider

$$\rho = (1-\lambda)\varphi + \lambda\tau$$

where $\tau = \frac{1}{d}$ as in the previous claim.

Note that ρ **is still a fractional perfect matching** such that

$$0 < \rho(e) < 1$$

on every edge. It does not lie on the extreme boundary of C_G, and it **can be distorted slightly at every edge** and still be in C_G.

Choose a **small** $\varepsilon < \lambda$.

Circuits

Use the previous lemma to find families of cycles $C_1, \ldots, C_K$ for $L = L(\varphi)$.

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For each $i \leq K$ consider the function $\zeta_i : \bigcup C_i \to \{\pm\varepsilon\}$ which **alternates** $\pm\varepsilon$ on the edges of (necessarily even) cycles in C_i

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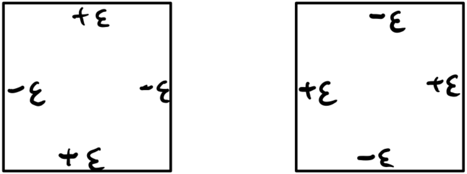

Random circuits

Consider **independent identically distributed** (iid) **random variables**:

$$Z_1(t), Z_2(t) \ldots \in \{-1, 1\}.$$

(for example for $t \in \{-1,1\}^{\mathbb{N}}$ let $Z_i(t) = t(i)$).

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(for example for $t \in \{-1,1\}^{\mathbb{N}}$ let $Z_i(t) = t(i)$).

For every t consider the following **distorted fractional perfect matching**

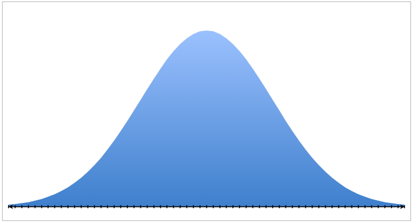
$$\rho_t = \rho + \sum_{i-1}^{K} Z_i(t)\zeta_i$$

Theorem (Berry–Esseen)

If $Y_1, Y_2 \ldots$ are iid with $\mathbb{E} Y_i = 0$, then

$$\lim_{k\to\infty} \mathbb{E}|\sum_{i=1}^{k} Y_i|/\sqrt{k} = \mathbb{E}|N| > 0,$$

where N has **normal distribution**.



Consequence

The latter implies that given K large enough, **for an edge** $e \in L(\varphi)$ we have

$$\mathbb{E}_t|\rho_t(e) - \frac{1}{2}| = \varepsilon \cdot \Omega(\sqrt{K})$$

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The latter implies that given K large enough, **for an edge** $e \in L(\varphi)$ we have

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On the other hand, **for an edge** $e \in G \setminus L(\varphi)$ we have $\varphi(e) \in \{0,1\}$ and the **distortion** $|\rho_t(e) - \varphi(e)|$ **is small**

$$|\rho_t(e) - \frac{1}{2}| > \frac{1}{2} - 2\lambda$$

Expected distortion

By Fubini's theorem, we get that in expected value:

$$\int_{E(G)} |\varphi(e) - \frac{1}{2}| d\mu < \mathbb{E}_t \int_{E(G)} |\rho_{\mathbf{t}}(e) - \frac{1}{2}| d\mu.$$

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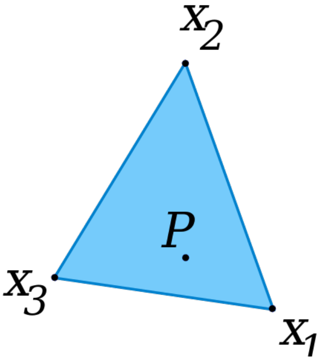
Find a witness

Since this is a convex condition, we can **find** t_0 such that

$$\int_{E(G)} |\varphi(e) - \frac{1}{2}| d\mu < \int_{E(G)} |\rho_{\mathbf{t_0}}(e) - \frac{1}{2}| d\mu.$$

Theorem (Choquet–Bishop–de Leeuw)

Each element of a compact convex set is a **barycenter of a probability measure** supported by the set of extreme points.



Applying this to ρ_{t_0}, we can **find an extreme point** ψ **which satisfies the same property as** ρ_{t_0}, i.e.

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This implies that $\mu(L(\psi)) < \mu(L(\varphi))$ and ends the proof of the lemma.

Limit construction

To get a perfect matching, we apply the above lemma a countable number of times.

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For **countable ordinals** α **we construct extreme points** φ_α of the set of fractional perfect matchings such that

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After **countably many times we get** $\mu(L(\varphi_\alpha)) = 0$ and φ_α is then a measurable perfect matching.

More general version

The proof does not use regularity in an essential way and also proves the following slightly more general version.

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Therem (BKS)

If a bipartite hyperfinite one-ended graphing admits a **measurable fractional perfect matching which is everywhere positive**, then it admits **measurable perfect matching**.

A further slightly more general version

Given a function $f : V(G) \to \mathbb{Z}$, a **fractional perfect f-matching** in a graph G is a function $\varphi : E(G) \to [0,1]$ such that

$$\sum_{y \in N_G(x)} \varphi(y) = f(x)$$

for every $x \in V(G)$.

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for every $x \in V(G)$.

Theorem (BKS)

Given a measurable function $f : V \to \mathbb{Z}$ If a bipartite hyperfinite one-ended graphing admits a **measurable fractional perfect f-matching** which is everywhere positive and bounded by c, then it admits **an integer-valued measurable fractional perfect f-matching** bounded by c.

Schreier graphings

Note that any **Schreier graphing of a group is regular** (r-regular when r is the size of the symmetric generating set).

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Bernoulli shifts

The **Bernoulli shift** of a group Γ is the action

$$\Gamma \curvearrowright [0,1]^\Gamma$$

by shift: $\gamma \cdot x(\delta) = x(\gamma^{-1}\delta)$.

Marked groups

By a **marked group** (Γ, S) we mean a finitely generated grop Γ with a fixed set S of generators.

Cayley graphs

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From the point of graph theory, a marked group is the same as its **Cayley graph**

Bernoulli graphing

Given marked group, we consider the **Schreier graphing of the Bernoulli shift**.

Factor of iid perfect matching

A **factor of iid perfect matching** of a marked group is a measurable perfect matching in the Bernoulli graphing.

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A **factor of iid perfect matching** of a marked group is a measurable perfect matching in the Bernoulli graphing.

Equivalently, a factor of iid perfect matching of a Cayley graph G can be defined as a **probability measure on the set of all perfect matchings** on G, which is a **factor of the product measure** on $[0,1]^{\Gamma}$.

Factor probability measure

Given two actions $\Gamma \curvearrowright (V_1, \nu_1)$ and $\Gamma \curvearrowright (V_2, \nu_2)$ the measure ν_2 is a **factor** of ν_1 is there exists a Γ-**invariant**

$$f : V_1 \to V_2$$

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In case of a factor iid of perfect matching on a Cayley graph, we consider the natural action of Γ **on the set of perfect matchings by left multiplication**.

Theorem (Lyons–Nazarov)

For any **nonamenable finitely generated group** Γ, any bipartite Cayley graph of Γ **has a factor of iid perfect matching**.

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Question (Lyons–Nazarov)

Which Cayley graphs admit a factor of iid perfect matching?

Corollary (to the perfect matching theorem)

Any bipartite Cayley graph of a **one-ended amenable group admits a factor of iid perfect matching**.

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Theorem (Bowen–Kun–S.)

A two-ended group admits a **factor of iid perfect matching** if and only if it is **not isomorphic to** $\mathbb{Z} \ltimes \Delta$ **with** Δ **finite of odd order**.

Corollary

- if Γ is **isomorphic to** $\mathbb{Z} \ltimes \Delta$ **with** $|\Delta|$ **odd**, then every bipartite Cayley graph of Γ **does not admit a factor of iid perfect matching**
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Perfect matchings have applications also in **equidecompositions**.

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Given an action $\Gamma \curvearrowright X$, two sets $A, B \subseteq X$ are **equidecomposable** if A can be partitioned as $\bigcup_{i=1}^n A_i$ such that B is partitioned as $B = \bigcup_{i=1}^n \gamma_i A_i$ for some $\gamma_i \in \Gamma$.

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Assuming the sets A and B are disjoint, A **and** B **are equidecomposable** using elements from a finite generating subset $S \subseteq \Gamma$

if and only if

the **bipartite Schreier graphing induced on** $A \cup B$ **has a perfect matching**.

Theorem (Laczkovich)

Cicrle squaring is possible, i.e. the **unit disc and the unit square on the plane are equidecomposable by translations**.
The same holds for any $A, B \subseteq \mathbb{R}^n$ of the same positive measure and $\dim_{\text{box}}(\partial A) < n$, $\dim_{\text{box}}(\partial B) < n$

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Theorem (Grabowski–Máthé–Pikhurko)

Measurable circle squaring is possible, i.e. the **unit disc and the unit square on the plane are equidecomposable by translations, using measurable pieces**.
The same holds for any $A, B \subseteq \mathbb{R}^n$ of the same positive measure and $\dim_{\text{box}}(\partial A) < n$, $\dim_{\text{box}}(\partial B) < n$

Corollary (to the perfect matching theorem)

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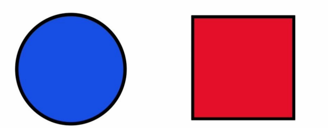

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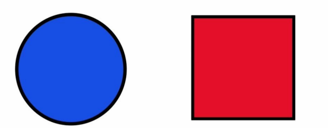

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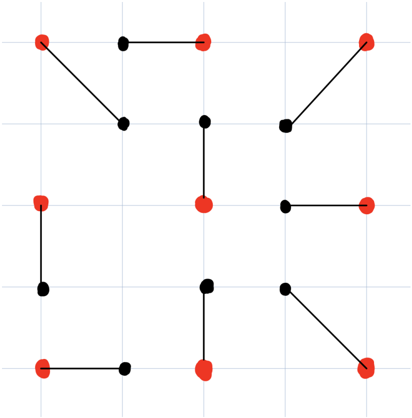
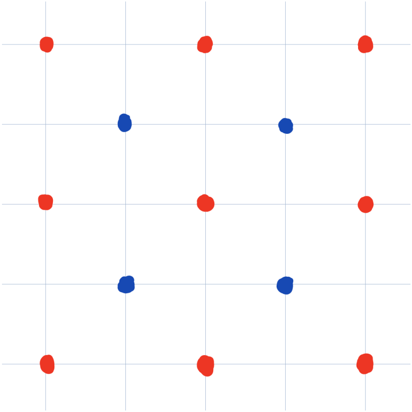
The group used in circle squaring is always $\mathbb{Z}^d$ for $d \gg 1$. The Schreier graphing is thus **hyperfinite and one-ended**.

Definition

A subset $A \subseteq \mathbb{R}^d$ is uniformly spread (with density α) if there is a bijection $f : A \to \frac{1}{\sqrt[d]{\alpha}}\mathbb{Z}^d$ such that $\sup_{x \in A} |f(x) - x| < \infty$.

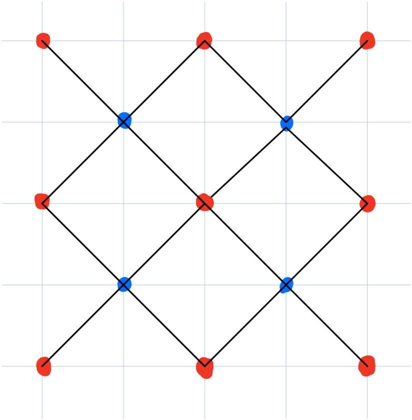


The action of $\mathbb{Z}^d$ is such that both sets are **uniformly spread**



Toast

The bipartite graphing can be approximated by a regular graphing coming from the distance graph on $\frac{1}{\sqrt[d]{\alpha}}\mathbb{Z}^d \cup (\frac{1}{\sqrt[d]{\alpha}}\mathbb{Z}^d + (1,\ldots,1))$



Positive fractional perfect matching

From this one can easily construct a measurable fractional perfect matching which is **positive on a one-ended set of edges**.

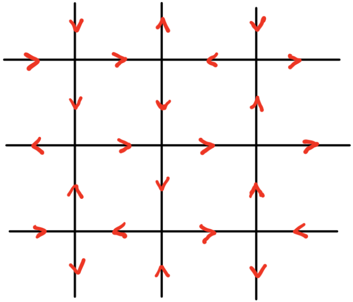
Corollary

The bipartite restriction of the Schreier graphing to the union of **disjoint copies circle and the square** admits a **measurable perfect matching**.

Balanced orientations

Given a $2r$-regular graph G, a **balanced orientation** of G is an assignment of orientations to the edges such that for every vertex x we have

$$\text{in-deg}(x) = \text{out-deg}(x)$$



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For Cayley graphs, it is simply a measurable balanced orientation of the Bernoulli shift.

Theorem (Bencs, Hrušková, Tóth)

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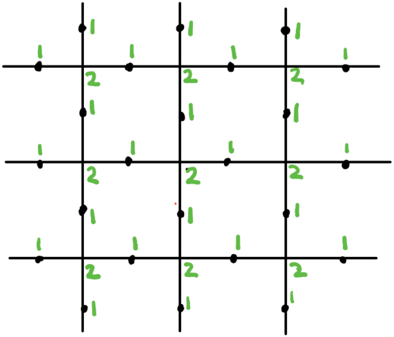
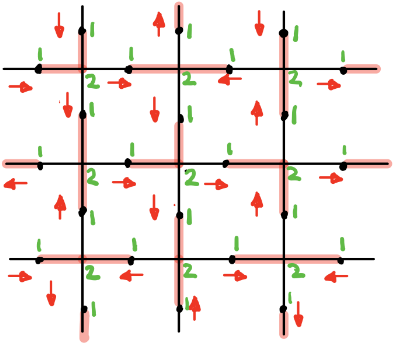
Does there exist a vertex-transitive graph that is not quasi-isometric to $\mathbb{Z}$ and has no factor of iid balanced orientation?

The perfect matching theorem can be used to answer this question in the negative.

Given a graph $2r$-regular graph G consider its **barycentric subdivision** G' and let $f : V(G') \to \mathbb{N}$ be 1 on the new vertices and r on $V(G)$.

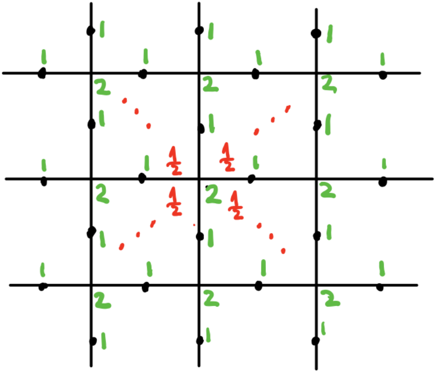


Any **perfect f-matching** in G' gives a **balanced orientation**:



Fractional perfect f-matching

It is easy to see that G' admits a **positive fractional perfect f-matching**.



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Corollary (to the perfect matching theorem)

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