KPT for weak Ramsey categories

Adam Bartoš

Institute of Mathematics, Czech Academy of Sciences

Winter School in Abstract Analysis
Section Set Theory & Topology
Hejnice, Jan 29 – Feb 5 2022

This is joint work with
Tristan Bice, Keegan Dasilva Barbosa, Wiesław Kubiś.

[1] A. Bartoš, T. Bice, K. Dasilva Barbosa, W. Kubiś, *The weak Ramsey property and extreme amenability*, arXiv:2110.01694.

Fraïssé theory

Theorem

Let $\mathcal{K}$ be a category

Fraïssé theory

Theorem

Let $\mathcal{K}$ be a category and let $\mathcal{L}$ be a free completion of $\mathcal{K}$.

Fraïssé theory

Theorem

Let $\mathcal{K}$ be a category and let $\mathcal{L}$ be a free completion of $\mathcal{K}$.
There exists a cofinal homogeneous object U in $\langle \mathcal{K}, \mathcal{L} \rangle$

Fraïssé theory

Theorem

Let $\mathcal{K}$ be a category and let $\mathcal{L}$ be a free completion of $\mathcal{K}$.
There exists a cofinal homogeneous object U in $\langle \mathcal{K}, \mathcal{L} \rangle$ if and only if $\mathcal{K}$ is a *Fraïssé category*, i.e.

Fraïssé theory

Theorem

Let $\mathcal{K}$ be a category and let $\mathcal{L}$ be a free completion of $\mathcal{K}$.
There exists a cofinal homogeneous object U in $\langle \mathcal{K}, \mathcal{L} \rangle$ if and only if $\mathcal{K}$ is a *Fraïssé category*, i.e.

1. $\mathcal{K}$ is directed,

Fraïssé theory

Theorem

Let $\mathcal{K}$ be a category and let $\mathcal{L}$ be a free completion of $\mathcal{K}$.
There exists a cofinal homogeneous object U in $\langle \mathcal{K}, \mathcal{L} \rangle$ if and only if $\mathcal{K}$ is a *Fraïssé category*, i.e.

1. $\mathcal{K}$ is directed,
2. $\mathcal{K}$ has the amalgamation property,

Fraïssé theory

Theorem

Let $\mathcal{K}$ be a category and let $\mathcal{L}$ be a free completion of $\mathcal{K}$.
There exists a cofinal homogeneous object U in $\langle \mathcal{K}, \mathcal{L} \rangle$ if and only if $\mathcal{K}$ is a *Fraïssé category*, i.e.

1. $\mathcal{K}$ is directed,
2. $\mathcal{K}$ has the amalgamation property,
3. $\mathcal{K}$ is dominated by a countable subcategory.

Fraïssé theory

Theorem

Let $\mathcal{K}$ be a category and let $\mathcal{L}$ be a free completion of $\mathcal{K}$.
There exists a cofinal homogeneous object U in $\langle \mathcal{K}, \mathcal{L} \rangle$ if and only if $\mathcal{K}$ is a *Fraïssé category*, i.e.

1. $\mathcal{K}$ is directed,
2. $\mathcal{K}$ has the amalgamation property,
3. $\mathcal{K}$ is dominated by a countable subcategory.

Such object U is unique and cofinal in $\mathcal{L}$. It is called the *Fraïssé limit*.

Fraïssé theory

Theorem

Let $\mathcal{K}$ be a category and let $\mathcal{L}$ be a free completion of $\mathcal{K}$.
There exists a cofinal homogeneous object U in $\langle \mathcal{K}, \mathcal{L} \rangle$ if and only if $\mathcal{K}$ is a *Fraïssé category*, i.e.

1. $\mathcal{K}$ is directed,
2. $\mathcal{K}$ has the amalgamation property,
3. $\mathcal{K}$ is dominated by a countable subcategory.

Such object U is unique and cofinal in $\mathcal{L}$. It is called the *Fraïssé limit*.

Genealogy: Fraïssé (countable structures), Droste and Göbel (categories), Kubiś (domination, Fraïssé sequences), B. (free completion)

Fraïssé theory

Examples

$\mathcal{K}$	$\mathcal{L}$	U

Fraïssé theory

Examples

	$\mathcal{K}$	$\mathcal{L}$	U
embeddings	finite linear orders	countable linear orders	the rationals

Fraïssé theory

Examples

	$\mathcal{K}$	$\mathcal{L}$	U
embeddings	finite linear orders	countable linear orders	the rationals
	finite graphs	countable graphs	Rado/random graph

Fraïssé theory

Examples

	$\mathcal{K}$	$\mathcal{L}$	U
embeddings	finite linear orders	countable linear orders	the rationals
	finite graphs	countable graphs	Rado/random graph
	finite groups	locally finite countable groups	Hall's universal group

Examples

	$\mathcal{K}$	$\mathcal{L}$	U
embeddings	finite linear orders	countable linear orders	the rationals
	finite graphs	countable graphs	Rado/random graph
	finite groups	locally finite countable groups	Hall's universal group
	finite rational metric spaces	countable rational metric spaces	rational Urysohn space

Examples

	$\mathcal{K}$	$\mathcal{L}$	U
embeddings	finite linear orders	countable linear orders	the rationals
	finite graphs	countable graphs	Rado/random graph
	finite groups	locally finite countable groups	Hall's universal group
	finite rational metric spaces	countable rational metric spaces	rational Urysohn space
quotients	finite discrete spaces	zero-dimensional metrizable compacta	Cantor space

Examples

	$\mathcal{K}$	$\mathcal{L}$	U
embeddings	finite linear orders	countable linear orders	the rationals
	finite graphs	countable graphs	Rado/random graph
	finite groups	locally finite countable groups	Hall's universal group
	finite rational metric spaces	countable rational metric spaces	rational Urysohn space
quotients	finite discrete spaces	zero-dimensional metrizable compacta	Cantor space
	finite discrete linear graphs	...	pseudoarc prespace [Irwin–Solecki]

Weak amalgamation property

Let $\mathcal{K}$ be a category and let $\alpha\colon a \to a'$ be a $\mathcal{K}$-map. We put $\mathcal{K}(\alpha, b) := \mathcal{K}(a', b) \circ \alpha = \{f \in \mathcal{K}(a, b) : f \text{ factorizes through } \alpha\}$.

Weak amalgamation property

Let $\mathcal{K}$ be a category and let $\alpha\colon a \to a'$ be a $\mathcal{K}$-map. We put $\mathcal{K}(\alpha, b) := \mathcal{K}(a', b) \circ \alpha = \{f \in \mathcal{K}(a, b) : f \text{ factorizes through } \alpha\}$.

Definition

A $\mathcal{K}$-map $\alpha\colon a \to a'$ is amalgamable if for every $f \in \mathcal{K}(\alpha, b)$ and $g \in \mathcal{K}(\alpha, c)$ there are $f', g' \in \mathcal{K}$ such that $f' \circ f = g' \circ g$.

Weak amalgamation property

Let $\mathcal{K}$ be a category and let $\alpha\colon a \to a'$ be a $\mathcal{K}$-map. We put $\mathcal{K}(\alpha, b) := \mathcal{K}(a', b) \circ \alpha = \{f \in \mathcal{K}(a, b) : f \text{ factorizes through } \alpha\}$.

Definition

A $\mathcal{K}$-map $\alpha\colon a \to a'$ is amalgamable if for every $f \in \mathcal{K}(\alpha, b)$ and $g \in \mathcal{K}(\alpha, c)$ there are $f', g' \in \mathcal{K}$ such that $f' \circ f = g' \circ g$.
$\mathcal{K}$ has the *weak amalgamation property* if for every $a \in \mathrm{Ob}(\mathcal{K})$ there is an amalgamable map $\alpha\colon a \to a'$.

Weak amalgamation property

Let $\mathcal{K}$ be a category and let $\alpha\colon a \to a'$ be a $\mathcal{K}$-map. We put $\mathcal{K}(\alpha, b) := \mathcal{K}(a', b) \circ \alpha = \{f \in \mathcal{K}(a, b) : f \text{ factorizes through } \alpha\}$.

Definition

A $\mathcal{K}$-map $\alpha\colon a \to a'$ is amalgamable if for every $f \in \mathcal{K}(\alpha, b)$ and $g \in \mathcal{K}(\alpha, c)$ there are $f', g' \in \mathcal{K}$ such that $f' \circ f = g' \circ g$.
$\mathcal{K}$ has the *weak amalgamation property* if for every $a \in \mathrm{Ob}(\mathcal{K})$ there is an amalgamable map $\alpha\colon a \to a'$.

- If α is amalgamable, then $\alpha \circ \beta$ and $\gamma \circ \alpha$ are amalgamable.

Weak amalgamation property

Let $\mathcal{K}$ be a category and let $\alpha\colon a \to a'$ be a $\mathcal{K}$-map. We put $\mathcal{K}(\alpha, b) := \mathcal{K}(a', b) \circ \alpha = \{f \in \mathcal{K}(a, b) : f \text{ factorizes through } \alpha\}$.

Definition

A $\mathcal{K}$-map $\alpha\colon a \to a'$ is amalgamable if for every $f \in \mathcal{K}(\alpha, b)$ and $g \in \mathcal{K}(\alpha, c)$ there are $f', g' \in \mathcal{K}$ such that $f' \circ f = g' \circ g$.
$\mathcal{K}$ has the *weak amalgamation property* if for every $a \in \mathrm{Ob}(\mathcal{K})$ there is an amalgamable map $\alpha\colon a \to a'$.

- If α is amalgamable, then $\alpha \circ \beta$ and $\gamma \circ \alpha$ are amalgamable.
- If $\mathcal{C} \subseteq \mathcal{K}$ is a full cofinal subcategory, then a $\mathcal{C}$-map α is amalgamable in $\mathcal{C}$ if and only if α is amalgamable in $\mathcal{K}$.

Weak Fraïssé theory

Theorem (Kubiś)

Let $\mathcal{K}$ be a category and let $\mathcal{L}$ be a free completion of $\mathcal{K}$.
There exists a cofinal weakly homogeneous object U in $\langle \mathcal{K}, \mathcal{L} \rangle$ if and only if $\mathcal{K}$ is a *weak Fraïssé category*, i.e.

1. $\mathcal{K}$ is directed,
2. $\mathcal{K}$ has the weak amalgamation property,
3. $\mathcal{K}$ is weakly dominated by a countable subcategory.

Such object U is unique and cofinal for $\mathcal{L}$-objects that are limits of $\mathcal{K}$-sequences consisting of amalgamable maps.

Weak Fraïssé theory

Theorem (Kubiś)

Let $\mathcal{K}$ be a category and let $\mathcal{L}$ be a free completion of $\mathcal{K}$. There exists a cofinal weakly homogeneous object U in $\langle \mathcal{K}, \mathcal{L} \rangle$ if and only if $\mathcal{K}$ is a *weak Fraïssé category*, i.e.

1. $\mathcal{K}$ is directed,
2. $\mathcal{K}$ has the weak amalgamation property,
3. $\mathcal{K}$ is weakly dominated by a countable subcategory.

Such object U is unique and cofinal for $\mathcal{L}$-objects that are limits of $\mathcal{K}$-sequences consisting of amalgamable maps.

Proposition

Let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{K}$ be full cofinal subcategories. Then $\mathcal{C}$ is weak Fraïssé if and only if $\mathcal{D}$ is weak Fraïssé, and they have the same Fraïssé limit.

Example – strong trees

Let $M \subseteq \mathbb{N}_+$ and let $\mathcal{T}_M$ denote the following category:

Example – strong trees

Let $M \subseteq \mathbb{N}_+$ and let $\mathcal{T}_M$ denote the following category:

- Objects are finite trees T endowed with a lexicographic order ("from left to right") such that for every non-terminal node $t \in T$ the splitting degree $\mathrm{spl}(t) \in M$.

Example – strong trees

Let $M \subseteq \mathbb{N}_+$ and let $\mathcal{T}_M$ denote the following category:

- Objects are finite trees T endowed with a lexicographic order ("from left to right") such that for every non-terminal node $t \in T$ the splitting degree $\mathrm{spl}(t) \in M$.
- Morphisms are embeddings preserving the meets, the lexicographic order, and the splitting degrees.

Example – strong trees

Let $M \subseteq \mathbb{N}_+$ and let $\mathcal{T}_M$ denote the following category:

- Objects are finite trees T endowed with a lexicographic order ("from left to right") such that for every non-terminal node $t \in T$ the splitting degree $\mathrm{spl}(t) \in M$.
- Morphisms are embeddings preserving the meets, the lexicographic order, and the splitting degrees.

Let $\mathcal{D}_M$ denote the modification of $\mathcal{T}_M$ where the splitting degree is preassigned also to every terminal node (and is preserved by the embeddings).

Example – strong trees

Let $M \subseteq \mathbb{N}_+$ and let $\mathcal{T}_M$ denote the following category:

- Objects are finite trees T endowed with a lexicographic order ("from left to right") such that for every non-terminal node $t \in T$ the splitting degree $\mathrm{spl}(t) \in M$.
- Morphisms are embeddings preserving the meets, the lexicographic order, and the splitting degrees.

Let $\mathcal{D}_M$ denote the modification of $\mathcal{T}_M$ where the splitting degree is preassigned also to every terminal node (and is preserved by the embeddings).

Proposition

$\mathcal{T}_M$ and $\mathcal{D}_M$ are full cofinal subcategories of a common supercategory. $\mathcal{T}_M$ is weak Fraïssé, while $\mathcal{D}_M$ is Fraïssé. $\mathcal{T}_M$ and $\mathcal{D}_M$ have a common Fraïssé limit U_M – a certain universal countable tree with branches isomorphic to $\mathbb{Q}$, related to the universal Ważewski dendrite W_{M+1}.

(Weak) Ramsey property

Let $\mathcal{K}$ be a category, $a, b, c \in \mathrm{Ob}(\mathcal{K})$, $k \in \omega$, $F \subseteq \mathcal{K}(a, b)$ finite.

(Weak) Ramsey property

Let $\mathcal{K}$ be a category, $a, b, c \in \mathrm{Ob}(\mathcal{K})$, $k \in \omega$, $F \subseteq \mathcal{K}(a, b)$ finite.

- $c \to (b)^a_{k,F}$ means that for every coloring $\varphi\colon \mathcal{K}(a, c) \to k$ there is $f\colon b \to c$ such that φ is constant on $f \circ F$.

(Weak) Ramsey property

Let $\mathcal{K}$ be a category, $a, b, c \in \mathrm{Ob}(\mathcal{K})$, $k \in \omega$, $F \subseteq \mathcal{K}(a, b)$ finite.

- $c \to (b)^a_{k,F}$ means that for every coloring $\varphi\colon \mathcal{K}(a, c) \to k$ there is $f\colon b \to c$ such that φ is constant on $f \circ F$.
- $\mathcal{K}$ has the *Ramsey property* if $(\forall a, b, k, F)(\exists c)\ c \to (b)^a_{k,F}$.

(Weak) Ramsey property

Let $\mathcal{K}$ be a category, $a, b, c \in \mathrm{Ob}(\mathcal{K})$, $k \in \omega$, $F \subseteq \mathcal{K}(a, b)$ finite.

- $c \to (b)^a_{k,F}$ means that for every coloring $\varphi \colon \mathcal{K}(a, c) \to k$ there is $f \colon b \to c$ such that φ is constant on $f \circ F$.
- $\mathcal{K}$ has the *Ramsey property* if $(\forall a, b, k, F)(\exists c)\ c \to (b)^a_{k,F}$.

Definition

A $\mathcal{K}$-map $\alpha \colon a \to a'$ is Ramsey if $(\forall b, k, F)(\exists c)\ c \to (b)^\alpha_{k,F}$ (where $c \to (b)^\alpha_{k,F}$ concerned with colorings $\varphi \colon \mathcal{K}(\alpha, c) \to k$).

(Weak) Ramsey property

Let $\mathcal{K}$ be a category, $a, b, c \in \mathrm{Ob}(\mathcal{K})$, $k \in \omega$, $F \subseteq \mathcal{K}(a, b)$ finite.

- $c \to (b)^a_{k,F}$ means that for every coloring $\varphi\colon \mathcal{K}(a, c) \to k$ there is $f\colon b \to c$ such that φ is constant on $f \circ F$.
- $\mathcal{K}$ has the *Ramsey property* if $(\forall a, b, k, F)(\exists c)\ c \to (b)^a_{k,F}$.

Definition

A $\mathcal{K}$-map $\alpha\colon a \to a'$ is Ramsey if $(\forall b, k, F)(\exists c)\ c \to (b)^\alpha_{k,F}$
(where $c \to (b)^\alpha_{k,F}$ concerned with colorings $\varphi\colon \mathcal{K}(\alpha, c) \to k$).
$\mathcal{K}$ has the *weak Ramsey property* if for every $a \in \mathrm{Ob}(\mathcal{K})$ there is a Ramsey map $\alpha\colon a \to a'$.

(Weak) Ramsey property

Let $\mathcal{K}$ be a category, $a, b, c \in \mathrm{Ob}(\mathcal{K})$, $k \in \omega$, $F \subseteq \mathcal{K}(a, b)$ finite.

- $c \to (b)^a_{k,F}$ means that for every coloring $\varphi\colon \mathcal{K}(a, c) \to k$ there is $f\colon b \to c$ such that φ is constant on $f \circ F$.
- $\mathcal{K}$ has the *Ramsey property* if $(\forall a, b, k, F)(\exists c)\ c \to (b)^a_{k,F}$.

Definition

A $\mathcal{K}$-map $\alpha\colon a \to a'$ is Ramsey if $(\forall b, k, F)(\exists c)\ c \to (b)^\alpha_{k,F}$
(where $c \to (b)^\alpha_{k,F}$ concerned with colorings $\varphi\colon \mathcal{K}(\alpha, c) \to k$).
$\mathcal{K}$ has the *weak Ramsey property* if for every $a \in \mathrm{Ob}(\mathcal{K})$ there is a Ramsey map $\alpha\colon a \to a'$.

- If α is Ramsey, then $\alpha \circ \beta$ and $\gamma \circ \alpha$ are Ramsey.

(Weak) Ramsey property

Let $\mathcal{K}$ be a category, $a, b, c \in \mathrm{Ob}(\mathcal{K})$, $k \in \omega$, $F \subseteq \mathcal{K}(a, b)$ finite.

- $c \to (b)^a_{k,F}$ means that for every coloring $\varphi\colon \mathcal{K}(a, c) \to k$ there is $f\colon b \to c$ such that φ is constant on $f \circ F$.
- $\mathcal{K}$ has the *Ramsey property* if $(\forall a, b, k, F)(\exists c)\ c \to (b)^a_{k,F}$.

Definition

A $\mathcal{K}$-map $\alpha\colon a \to a'$ is Ramsey if $(\forall b, k, F)(\exists c)\ c \to (b)^\alpha_{k,F}$
(where $c \to (b)^\alpha_{k,F}$ concerned with colorings $\varphi\colon \mathcal{K}(\alpha, c) \to k$).
$\mathcal{K}$ has the *weak Ramsey property* if for every $a \in \mathrm{Ob}(\mathcal{K})$ there is a Ramsey map $\alpha\colon a \to a'$.

- If α is Ramsey, then $\alpha \circ \beta$ and $\gamma \circ \alpha$ are Ramsey.
- If $\mathcal{K}$ is directed and α is Ramsey, then α is amalgamable.

(Weak) Ramsey property

Let $\mathcal{K}$ be a category, $a, b, c \in \mathrm{Ob}(\mathcal{K})$, $k \in \omega$, $F \subseteq \mathcal{K}(a, b)$ finite.

- $c \to (b)^a_{k,F}$ means that for every coloring $\varphi\colon \mathcal{K}(a, c) \to k$ there is $f\colon b \to c$ such that φ is constant on $f \circ F$.
- $\mathcal{K}$ has the *Ramsey property* if $(\forall a, b, k, F)(\exists c)\ c \to (b)^a_{k,F}$.

Definition

A $\mathcal{K}$-map $\alpha\colon a \to a'$ is Ramsey if $(\forall b, k, F)(\exists c)\ c \to (b)^\alpha_{k,F}$
(where $c \to (b)^\alpha_{k,F}$ concerned with colorings $\varphi\colon \mathcal{K}(\alpha, c) \to k$).
$\mathcal{K}$ has the *weak Ramsey property* if for every $a \in \mathrm{Ob}(\mathcal{K})$ there is a Ramsey map $\alpha\colon a \to a'$.

- If α is Ramsey, then $\alpha \circ \beta$ and $\gamma \circ \alpha$ are Ramsey.
- If $\mathcal{K}$ is directed and α is Ramsey, then α is amalgamable.
- If $\mathcal{K}$ has WRP and α is amalgamable, then α is Ramsey.

(Weak) Ramsey property

Let $\mathcal{K}$ be a category, $a, b, c \in \mathrm{Ob}(\mathcal{K})$, $k \in \omega$, $F \subseteq \mathcal{K}(a, b)$ finite.

- $c \to (b)^a_{k,F}$ means that for every coloring $\varphi\colon \mathcal{K}(a, c) \to k$ there is $f\colon b \to c$ such that φ is constant on $f \circ F$.
- $\mathcal{K}$ has the *Ramsey property* if $(\forall a, b, k, F)(\exists c)\ c \to (b)^a_{k,F}$.

Definition

A $\mathcal{K}$-map $\alpha\colon a \to a'$ is Ramsey if $(\forall b, k, F)(\exists c)\ c \to (b)^\alpha_{k,F}$
(where $c \to (b)^\alpha_{k,F}$ concerned with colorings $\varphi\colon \mathcal{K}(\alpha, c) \to k$).
$\mathcal{K}$ has the *weak Ramsey property* if for every $a \in \mathrm{Ob}(\mathcal{K})$ there is a Ramsey map $\alpha\colon a \to a'$.

- If α is Ramsey, then $\alpha \circ \beta$ and $\gamma \circ \alpha$ are Ramsey.
- If $\mathcal{K}$ is directed and α is Ramsey, then α is amalgamable.
- If $\mathcal{K}$ has WRP and α is amalgamable, then α is Ramsey.
- So in a directed category we have RP $\iff$ WRP & AP.

(Weak) Ramsey property

Let $\mathcal{K}$ be a category, $a, b, c \in \mathrm{Ob}(\mathcal{K})$, $k \in \omega$, $F \subseteq \mathcal{K}(a, b)$ finite.

- $c \to (b)^a_{k,F}$ means that for every coloring $\varphi\colon \mathcal{K}(a, c) \to k$ there is $f\colon b \to c$ such that φ is constant on $f \circ F$.
- $\mathcal{K}$ has the *Ramsey property* if $(\forall a, b, k, F)(\exists c)\ c \to (b)^a_{k,F}$.

Definition

A $\mathcal{K}$-map $\alpha\colon a \to a'$ is Ramsey if $(\forall b, k, F)(\exists c)\ c \to (b)^\alpha_{k,F}$ (where $c \to (b)^\alpha_{k,F}$ concerned with colorings $\varphi\colon \mathcal{K}(\alpha, c) \to k$).
$\mathcal{K}$ has the *weak Ramsey property* if for every $a \in \mathrm{Ob}(\mathcal{K})$ there is a Ramsey map $\alpha\colon a \to a'$.

- If α is Ramsey, then $\alpha \circ \beta$ and $\gamma \circ \alpha$ are Ramsey.
- If $\mathcal{K}$ is directed and α is Ramsey, then α is amalgamable.
- If $\mathcal{K}$ has WRP and α is amalgamable, then α is Ramsey.
- So in a directed category we have RP $\iff$ WRP & AP.
- If $\mathcal{C} \subseteq \mathcal{K}$ is a full cofinal subcategory, then a $\mathcal{C}$-map α is Ramsey in $\mathcal{C}$ if and only if α is Ramsey in $\mathcal{K}$.

Topological dynamics

A continuous action $G \curvearrowright X$ of a topological group G on a topological space X is a continuous map $G \times X \to X$ such that the corresponding map $G \to X^X$ is a homomorphism.

Topological dynamics

A continuous action $G \curvearrowright X$ of a topological group G on a topological space X is a continuous map $G \times X \to X$ such that the corresponding map $G \to X^X$ is a homomorphism.

A topological group G is *extremely amenable* if every continuous action $G \curvearrowright X$ on a compact space has a fixed point.

Topological dynamics

A continuous action $G \curvearrowright X$ of a topological group G on a topological space X is a continuous map $G \times X \to X$ such that the corresponding map $G \to X^X$ is a homomorphism.

A topological group G is *extremely amenable* if every continuous action $G \curvearrowright X$ on a compact space has a fixed point.

Definition (Pestov)

An action $G \curvearrowright X$ on a discrete space is *finitely oscillation stable* if for every finite $F \subseteq X$, every $k \in \omega$, and every coloring $\varphi \colon X \to k$ there is $g \in G$ such that φ is constant on gF.

Topological dynamics

A continuous action $G \curvearrowright X$ of a topological group G on a topological space X is a continuous map $G \times X \to X$ such that the corresponding map $G \to X^X$ is a homomorphism.

A topological group G is *extremely amenable* if every continuous action $G \curvearrowright X$ on a compact space has a fixed point.

Definition (Pestov)

An action $G \curvearrowright X$ on a discrete space is *finitely oscillation stable* if for every finite $F \subseteq X$, every $k \in \omega$, and every coloring $\varphi \colon X \to k$ there is $g \in G$ such that φ is constant on gF.

Theorem (essentially Kechris–Pestov–Todorčević, 2005)

A topological group G with a neighborhood base $\mathcal{V}$ at the unit consisting of open subgroups is extremely amenable if and only if the actions $G \curvearrowright G/V$ for $V \in \mathcal{V}$ are finitely oscillation stable.

Topological dynamics

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and U an $\mathcal{L}$-object.

Topological dynamics

Let $\langle\mathcal{K},\mathcal{L}\rangle$ be a free completion and U an $\mathcal{L}$-object.

- $\mathrm{Aut}(U)$ has a completely metrizable topology whose base at identity consists of the stabilizers of the actions $\mathrm{Aut}(U) \curvearrowright \mathcal{L}(a, U)$, $a \in \mathrm{Ob}(\mathcal{K})$.

Topological dynamics

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and U an $\mathcal{L}$-object.

- $\mathrm{Aut}(U)$ has a completely metrizable topology whose base at identity consists of the stabilizers of the actions $\mathrm{Aut}(U) \curvearrowright \mathcal{L}(a, U)$, $a \in \mathrm{Ob}(\mathcal{K})$.
- The topology is separable (and so Polish) if and only if $\mathcal{K}$ is locally countable.

Topological dynamics

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and U an $\mathcal{L}$-object.

- $\mathrm{Aut}(U)$ has a completely metrizable topology whose base at identity consists of the stabilizers of the actions $\mathrm{Aut}(U) \curvearrowright \mathcal{L}(a, U)$, $a \in \mathrm{Ob}(\mathcal{K})$.
- The topology is separable (and so Polish) if and only if $\mathcal{K}$ is locally countable.
- In the classical case when U is a countable structure, this is the topology of pointwise convergence.

Topological dynamics

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and U an $\mathcal{L}$-object.

- $\mathrm{Aut}(U)$ has a completely metrizable topology whose base at identity consists of the stabilizers of the actions $\mathrm{Aut}(U) \curvearrowright \mathcal{L}(a, U)$, $a \in \mathrm{Ob}(\mathcal{K})$.
- The topology is separable (and so Polish) if and only if $\mathcal{K}$ is locally countable.
- In the classical case when U is a countable structure, this is the topology of pointwise convergence.

Theorem (B., Bice, Dasilva Barbosa, Kubiś)

If U is a cofinal weakly homogeneous object in a free completion $\langle \mathcal{K}, \mathcal{L} \rangle$, then a $\mathcal{K}$-map α is Ramsey if and only if the action $\mathrm{Aut}(U) \curvearrowright \mathcal{L}(\alpha, U)$ is finitely oscillation stable.

Kechris–Pestov–Todorčević correspondence

Theorem (B., Bice, Dasilva Barbosa, Kubiś)

If U is a cofinal weakly homogeneous object in a free completion $\langle \mathcal{K}, \mathcal{L} \rangle$, then the topological group $\mathrm{Aut}(U)$ is *extremely amenable* if and only if the category $\mathcal{K}$ has the *weak Ramsey property*.

Kechris–Pestov–Todorčević correspondence

Theorem (B., Bice, Dasilva Barbosa, Kubiś)

If U is a cofinal weakly homogeneous object in a free completion $\langle \mathcal{K}, \mathcal{L} \rangle$, then the topological group $\mathrm{Aut}(U)$ is *extremely amenable* if and only if the category $\mathcal{K}$ has the *weak Ramsey property*.

Classical KPT correspondence is concerned with first-order structures.

Kechris–Pestov–Todorčević correspondence

Theorem (B., Bice, Dasilva Barbosa, Kubiś)

If U is a cofinal weakly homogeneous object in a free completion $\langle \mathcal{K}, \mathcal{L} \rangle$, then the topological group $\mathrm{Aut}(U)$ is *extremely amenable* if and only if the category $\mathcal{K}$ has the *weak Ramsey property*.

Classical KPT correspondence is concerned with first-order structures. We generalize in two ways:

Kechris–Pestov–Todorčević correspondence

Theorem (B., Bice, Dasilva Barbosa, Kubiś)

If U is a cofinal weakly homogeneous object in a free completion $\langle \mathcal{K}, \mathcal{L} \rangle$, then the topological group $\mathrm{Aut}(U)$ is *extremely amenable* if and only if the category $\mathcal{K}$ has the *weak Ramsey property*.

Classical KPT correspondence is concerned with first-order structures. We generalize in two ways:

- from classes of structures to categories,

Kechris–Pestov–Todorčević correspondence

Theorem (B., Bice, Dasilva Barbosa, Kubiś)

If U is a cofinal weakly homogeneous object in a free completion $\langle \mathcal{K}, \mathcal{L} \rangle$, then the topological group $\mathrm{Aut}(U)$ is *extremely amenable* if and only if the category $\mathcal{K}$ has the *weak Ramsey property*.

Classical KPT correspondence is concerned with first-order structures. We generalize in two ways:

- from classes of structures to categories,
- from Fraïssé categories to weak Fraïssé categories.

Kechris–Pestov–Todorčević correspondence

Theorem (B., Bice, Dasilva Barbosa, Kubiś)

If U is a cofinal weakly homogeneous object in a free completion $\langle \mathcal{K}, \mathcal{L} \rangle$, then the topological group $\mathrm{Aut}(U)$ is *extremely amenable* if and only if the category $\mathcal{K}$ has the *weak Ramsey property*.

Classical KPT correspondence is concerned with first-order structures. We generalize in two ways:

- from classes of structures to categories,
- from Fraïssé categories to weak Fraïssé categories.

  (A weak Fraïssé category has the Ramsey property if and only if it has the weak Ramsey property and the amalgamation property.)

Example – strong trees (continued)

- For every $M \subseteq \mathbb{N}_+$ the weak Fraïssé category $\mathcal{T}_M$ has the weak Ramsey property (but not the Ramsey property since it has no amalgamable objects).

Example – strong trees (continued)

- For every $M \subseteq \mathbb{N}_+$ the weak Fraïssé category $\mathcal{T}_M$ has the weak Ramsey property (but not the Ramsey property since it has no amalgamable objects).
- This is because the Fraïssé category $\mathcal{D}_M$ has the Ramsey property (essentially proved by Kwiatkowska).

Example – strong trees (continued)

- For every $M \subseteq \mathbb{N}_+$ the weak Fraïssé category $\mathcal{T}_M$ has the weak Ramsey property (but not the Ramsey property since it has no amalgamable objects).
- This is because the Fraïssé category $\mathcal{D}_M$ has the Ramsey property (essentially proved by Kwiatkowska).
- It follows that the automorphism group of the universal tree U_M is extremely amenable.

Example – strong trees (continued)

- For every $M \subseteq \mathbb{N}_+$ the weak Fraïssé category $\mathcal{T}_M$ has the weak Ramsey property (but not the Ramsey property since it has no amalgamable objects).
- This is because the Fraïssé category $\mathcal{D}_M$ has the Ramsey property (essentially proved by Kwiatkowska).
- It follows that the automorphism group of the universal tree U_M is extremely amenable.

Question

Let $\mathcal{T}'_M$ and $\mathcal{D}'_M$ be the modified categories of strong trees where the levels are part of the structure and the embeddings are required to preserve them.

Example – strong trees (continued)

- For every $M \subseteq \mathbb{N}_+$ the weak Fraïssé category $\mathcal{T}_M$ has the weak Ramsey property (but not the Ramsey property since it has no amalgamable objects).
- This is because the Fraïssé category $\mathcal{D}_M$ has the Ramsey property (essentially proved by Kwiatkowska).
- It follows that the automorphism group of the universal tree U_M is extremely amenable.

Question

Let $\mathcal{T}'_M$ and $\mathcal{D}'_M$ be the modified categories of strong trees where the levels are part of the structure and the embeddings are required to preserve them. Does $\mathcal{T}'_M$ have the weak Ramsey property? Equivalently, does $\mathcal{D}'_M$ have the Ramsey property?

Example – strong trees (continued)

- For every $M \subseteq \mathbb{N}_+$ the weak Fraïssé category $\mathcal{T}_M$ has the weak Ramsey property (but not the Ramsey property since it has no amalgamable objects).
- This is because the Fraïssé category $\mathcal{D}_M$ has the Ramsey property (essentially proved by Kwiatkowska).
- It follows that the automorphism group of the universal tree U_M is extremely amenable.

Question

Let $\mathcal{T}'_M$ and $\mathcal{D}'_M$ be the modified categories of strong trees where the levels are part of the structure and the embeddings are required to preserve them. Does $\mathcal{T}'_M$ have the weak Ramsey property? Equivalently, does $\mathcal{D}'_M$ have the Ramsey property?

- We have shown that $\mathcal{T}'_M$ is weak Fraïssé and that $\mathcal{D}'_M$ is Fraïssé, and characterized their common Fraïssé limit.

Example – strong trees (continued)

- For every $M \subseteq \mathbb{N}_+$ the weak Fraïssé category $\mathcal{T}_M$ has the weak Ramsey property (but not the Ramsey property since it has no amalgamable objects).
- This is because the Fraïssé category $\mathcal{D}_M$ has the Ramsey property (essentially proved by Kwiatkowska).
- It follows that the automorphism group of the universal tree U_M is extremely amenable.

Question

Let $\mathcal{T}'_M$ and $\mathcal{D}'_M$ be the modified categories of strong trees where the levels are part of the structure and the embeddings are required to preserve them. Does $\mathcal{T}'_M$ have the weak Ramsey property? Equivalently, does $\mathcal{D}'_M$ have the Ramsey property?

- We have shown that $\mathcal{T}'_M$ is weak Fraïssé and that $\mathcal{D}'_M$ is Fraïssé, and characterized their common Fraïssé limit.
- The case $M = \{m\}$ follows from the Milliken's theorem.

Theorem (B., Bice, Dasilva Barbosa, Kubiś)

If U is a cofinal weakly homogeneous object in a free completion $\langle \mathcal{K}, \mathcal{L} \rangle$, then the topological group $\mathrm{Aut}(U)$ is *extremely amenable* if and only if the category $\mathcal{K}$ has the *(weak) Ramsey property*.

[1] A. Bartoš, T. Bice, K. Dasilva Barbosa, W. Kubiś, *The weak Ramsey property and extreme amenability*, arXiv:2110.01694.

[2] W. Kubiś, *Weak Fraïssé limits*, Theory Appl. Categ. 38 (2022).

[3] A.S. Kechris, V.G. Pestov, S. Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geom. Funct. Anal. 15 (2005).

[4] V. Pestov, *Dynamics of infinite-dimensional groups*, vol. 40 of University Lecture Series, American Mathematical Society, Providence, RI, 2006.

Thank you.