The tower spectrum

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The Spectrum Tower, Warsaw

Definition

A *tower* is a sequence $\langle x_\alpha : \alpha < \delta \rangle$ of infinite subsets of ω, such that

- $\forall \alpha < \beta < \delta (x_\beta \subseteq^* x_\alpha)$, where $x_\beta \subseteq^* x_\alpha$ iff $|x_\beta \setminus x_\alpha| < \omega$
- $\forall x \in [\omega]^\omega \exists \alpha < \delta (x \not\subseteq^* x_\alpha)$

Question

What is the least δ such that there is a tower of length δ?

The answer is the regular cardinal $\mathfrak{t}$. We will ask:

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For what δ is there a tower of length δ? More specifically: For which regular cardinals κ is there a tower of length κ?

The answer is the *tower spectrum* that we will denote with $\mathcal{T}$.

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$$\mathcal{T} := \{\kappa : \kappa \text{ regular and there is a tower of length } \kappa\}$$

Obviously $\mathcal{T} \subseteq [\aleph_1, 2^{\aleph_0}]$.
Main goal: control $\mathcal{T}$.
This has been done for mad families before:

Theorem (Blass; Shelah, Spinas)

(GCH) Let $\mathcal{C}$ be a set of uncountable cardinals so that

- *$\mathcal{C}$ is closed under singular limits,*
- *$\mathcal{C}$ has a maximum,*
- *$\max \mathcal{C}$ has uncountable cofinality,*
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Then there is a ccc forcing extension in which $\mathcal{A} = \mathcal{C}$.

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What is the main idea?

Hechler defined a ccc poset $\mathbb{H}_{\mathsf{mad}}(\kappa)$ for adding a mad family of size κ by finite approximations.

Given $\mathcal{C}$ as above we simply force with $\mathbb{P} := \prod^{<\omega}_{\kappa \in \mathcal{C}} \mathbb{H}_{\mathsf{mad}}(\kappa)$.

Using a modification of $\mathbb{H}_{\mathsf{mad}}(\kappa)$ adding a tower of length κ we could show the following:

Theorem (S.)

Assume there are infinitely many weakly compact cardinals. Let $C \subseteq \omega \setminus \{0\}$. Then there is a forcing extension in which for every $n \in \omega$,

$$\aleph_{2n} \in \mathcal{T} \leftrightarrow n \in C.$$

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Instead we have a **new idea**. Let us outline a general framework: Let L be a lattice with a top element $l_{top} \in L$ and δ an ordinal. Let $\{\mathbb{B}_l^\alpha : l \in L, \alpha \leq \delta\}$ be a set of complete boolean algebras such that $\mathbb{B}_l^\alpha \lessdot \mathbb{B}_k^\beta$ for $\alpha \leq \beta$ and $l \leq k$. Then we call this an amalgamation system if:

1. $\forall l \in L \forall \alpha \in \lim(\delta+1)(\mathbb{B}_l^\alpha = \varinjlim_{\beta<\alpha} \mathbb{B}_l^\beta)$,
2. $\forall l \in \lim L \forall \alpha \leq \delta(\mathbb{B}_l^\alpha = \varinjlim_{k<l} \mathbb{B}_k^\alpha)$[1]
3. $\forall k_0, k_1 \leq l \in L \forall \alpha, \beta \leq \gamma \leq \delta$

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Suppose that l_{top} is a limit, $L \setminus \{l_{top}\}$ is σ-directed, $\omega < \mathrm{cf}(\delta)$ and $\mathbb{B}_{l_{top}}^\delta$ is ccc. Then whenever $\dot{x}$ is a $\mathbb{B}_{l_{top}}^\delta$-name for a real, there is $l \in L \setminus \{l_{top}\}$, $\alpha < \delta$ and a $\mathbb{B}_l^\alpha$-name $\dot{y}$, such that $\Vdash \dot{x} = \dot{y}$.

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The actual construction (under GCH):

Given a lattice L as above and a regular cardinal $\lambda \geq |L|$ let $\{l_\alpha : \alpha < \lambda\}$ enumerate $L \setminus \{l_{\text{top}}\}$ such that every l appears λ many times.

- We start with $\mathbb{B}^0_l$ the trivial Boolean algebra for every $l \in L$.
- Suppose that $\mathbb{B}^\alpha_l$ has been defined for $\alpha < \gamma \leq \lambda$ and all $l \in L$, then:
  - γ limit: let $\mathbb{B}^\gamma_l = \varinjlim_{\alpha<\gamma} \mathbb{B}^\alpha_l$,
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$$\mathbb{B}^{\alpha+1}_l := \mathbb{B}^\alpha_l * \dot{\mathbb{Q}}_\alpha \text{ if } l_\alpha < l \text{ and}$$
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Note that this makes sense since by induction $\mathbb{B}^\alpha_{l_\alpha} \lessdot \mathbb{B}^\alpha_l$ for every $l > l_\alpha$.

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Given a lattice L as above and a regular cardinal $\lambda \geq |L|$ let $\{l_\alpha : \alpha < \lambda\}$ enumerate $L \setminus \{l_{\text{top}}\}$ such that every l appears λ many times.

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$\{\mathbb{B}_l^\alpha : \alpha \leq \lambda, l \in L\}$ is an amalgamation system of CBAs. Moreover $\mathbb{B}_{l_{\text{top}}}^\lambda$ has the ccc (and in particular all $\mathbb{B}_l^\alpha$'s).

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Definition

Let $X \subseteq L \setminus \{l_{\text{top}}\}$ then X is called κ-unbounded if $|X| = \kappa$ and for any $Y \subseteq X$,

$$Y \text{ is bounded } \rightarrow |Y| < \kappa.$$

Theorem

Assume that $\kappa < \lambda$ and there is no κ-unbounded subset in $L \setminus \{l_{\text{top}}\}$, then

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Suppose $\langle \dot{x}_\xi : \xi < \kappa \rangle$ is forced to be a tower. For each $\xi < \kappa$ we can assume that $\dot{x}_\xi$ is a $\mathbb{B}^{\alpha_\xi}_{k_\xi}$ name for some $\alpha_\xi < \lambda$ and $k_\xi \in L \setminus \{l_{\text{top}}\}$. As $\kappa < \lambda$ we have that $\sup_{\xi < \kappa} \alpha_\xi = \alpha < \lambda$. Moreover since there is no κ-unbounded subset of L, there is $X \in [\kappa]^\kappa$ so that $\{k_\xi : \xi \in X\}$ is bounded, say by $l \in L \setminus \{l_{\text{top}}\}$. Then $\langle x_\xi : \xi \in X \rangle$ is added by $\mathbb{B}^{\alpha}_{l}$.

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$\mathbb{B}^{\alpha_\xi}_{k_{\xi+1}} = \mathbb{B}^\beta_{k_{\xi+1}} * \dot{\mathbb{Q}}_\beta$ since $k_\xi < k_{\xi+1}$ and $\mathbb{B}^{\alpha_\xi}_{l \wedge k_{\xi+1}} = \mathbb{B}^\beta_{l \wedge k_{\xi+1}}$ since $k_\xi \not< l \wedge k_{\xi+1}$.
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In particular the real added by $\mathbb{Q}_\beta$, namely x_ξ, is going to be generic over $V^{\mathbb{B}^\alpha_l} \ni x$. This guarantees that $x \not\subseteq^* x_\xi$.

□

Proof.

(...) Claim: $\langle x_\xi : \xi < \kappa \rangle$ will be a tower.
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Theorem

Assume that $\kappa < \lambda$ *and there is no* κ*-unbounded subset in* $L \setminus \{l_{\text{top}}\}$*, then*

$$V^{\mathbb{B}^{\lambda}_{l_{\text{top}}}} \models \kappa \notin \mathcal{T}.$$

Theorem

Assume that there is a strictly increasing unbounded sequence $\langle k_\xi : \xi < \kappa \rangle$ *in* $L \setminus \{l_{\text{top}}\}$*, then*

$$V^{\mathbb{B}^{\lambda}_{l_{\text{top}}}} \models \kappa \in \mathcal{T}.$$

The actual actual construction:

Let $C \subseteq \omega \setminus \{0\}$ non-empty and consider the lattice $L = \prod_{n \in C} \aleph_n$ with $f \wedge g = \min(f, g)$. $|L|$ is regular uncountable (either $\aleph_{\max C}$ or $\aleph_{\omega+1}$). For any $n \in C$, L has a $\aleph_n$-length unbounded increasing sequence. If $n \notin C$ then L has no $\aleph_n$-unbounded set.
Thus:

Theorem (S.)

(GCH) Let $C \subseteq \omega \setminus \{0\}$. Then there is a ccc forcing notion $\mathbb{P}$ so that

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What about other spectra? The same method applies to:

- The set of κ such that there is a κ-filterbase on ω. $\mathcal{F}$ is a κ-filterbase if $|\mathcal{F}| = \kappa$ and $\forall \mathcal{A} \subseteq \mathcal{F}(\exists x(x \subseteq^* \mathcal{A}) \to |\mathcal{A}| < \kappa)$.
- The set of κ such that there is a κ-unbounded subset of $\omega^\omega / \text{fin}$.
- The lengths of "unbounded scales" in $\omega^\omega / \text{fin}$.
- The set of κ such that there is a κ-concentrated subset of $\mathbb{R}$.
- The set of κ such that there is a κ-Luzin set. $X \subseteq \mathbb{R}$ is a κ-Luzin set if $|X| = \kappa$ and $\forall Y \subseteq X(Y \text{ is meager } \leftrightarrow |Y| < \kappa)$.
- The lengths of eventually splitting sequences. $\langle x_\xi : \xi < \kappa \rangle$ is eventually splitting if $\forall x \in [\omega]^\omega \exists \xi < \kappa \forall \eta > \xi (x_\eta \text{ splits } x)$.
- ...find your own example!

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Thank you for your attention!