Countable and Uncountable Extremally Disconnected Groups and Related Ultrafilters

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All topological spaces are assumed to be completely regular and Hausdorff.

Problem (Arhangelskii, 1967)

Does there exist in ZFC *a nondiscrete extremally disconnected topological group?*

Definition

A topological space is said to be extremally disconnected if the closure of any open set in this space is open (or, equivalently, the closures of any two disjoint open sets are disjoint).

Frolík:

- ($\aleph_0^+ = 2^{\aleph_0}$ or $(2^{\aleph_0})^+ \neq 2^{2^{\aleph_0}}$) Any homogeneneous extremally disconnected compact space is finite. (NB: There exist nondiscrete infinite extremally disconnected spaces.)

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- (Malykhin) *Any extremally disconnected group contains an open Boolean subgroup.*

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- the set $[X]^{<\omega}$ of all finite subsets of X with the operation $\triangle$ of symmetric difference ($A \triangle B = A \cup B \setminus (A \cap B)$);
- any Boolean group with basis X is isomorphic to the direct sum $\bigoplus^{|X|} \mathbb{Z}_2$ of $|X|$ copies of $\mathbb{Z}_2$, i.e., the set of finitely supported maps $g \colon X \to \mathbb{Z}_2$ with pointwise addition (in the field $\mathbb{F}_2$).

Simplest extremally disconnected space:

Each free filter $\mathscr{F}$ on any set X is associated with $X_{\mathscr{F}} = X \cup \{*\}$ ($*$ is a point not belonging to X); all points of X are isolated and the neighborhoods of $*$ are $\{*\} \cup A$, $A \in \mathscr{F}$.

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Simplest candidate for an extremally disconnected group:

$B^{\mathrm{lin}}(X_{\mathscr{F}})$ is $B(X) = [X]^{<\omega}$ with the group topology generated by the neighborhood base at zero

$$U = \{\mathbf{a} \in [X]^{<\omega} : \mathbf{a} \subset A\} = \langle A \rangle, \qquad A \in \mathscr{F}$$

($\langle A \rangle$ is the subgroup generated by A).

The only nonizolated point $*$ of $X_{\mathscr{F}}$ is identified with $0 = \varnothing$, and each $x \in X$ is identified with $\{x\} \in [X]^{<\omega}$.

Sirota (1968):

- defined a selective ultrafilter on ω and proved its existence under CH;
- proved that if $\mathscr{U}$ is a selective ultrafilter on ω, then $B^{\mathrm{lin}}(\omega_{\mathscr{U}})$ is an extremally disconnected group.

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An ultrafilter on ω is selective if, for any partition $\{C_n : n \in \omega\}$ of ω such that $C_n \notin \mathscr{U}$ for $n \in \omega$, there exists a *selector* in $\mathscr{U}$, that is, a set $A \in \mathscr{U}$ such that $|A \cap C_n| = 1$ for all n.

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Ramsey's theorem: If $n \in \mathbb{N}$ and the set $[\omega]^n$ of n-element subsets of ω is partitioned into finitely many pieces, then there is an infinite set $H \subset \omega$ homogeneous with respect to this partition, i.e., such that $[H]^n$ is contained in one of the pieces.

An ultrafilter $\mathscr{U}$ on ω is called a Ramsey ultrafilter if, given any positive integers n and k, every partition $F\colon [\omega]^n \to \{1, \ldots, k\}$ has a homogeneous set $H \in \mathscr{U}$.

Theorem (Booth+Kunen)

The following conditions on a free ultrafilter $\mathscr{U}$ on ω are equivalent:

(i) $\mathscr{U}$ *is Ramsey;*

(ii) $\mathscr{U}$ *is selective: for any partition $\{C_n : n \in \omega\}$ of ω such that $C_n \notin \mathscr{U}$ for $n \in \omega$, there exists a selector in $\mathscr{U}$, that is, a set $A \in \mathscr{U}$ such that $|A \cap C_n| = 1$ for all n;*

(iii) *for any sequence $\{A_n : n \in \omega\}$, where $A_n \in \mathscr{U}$, there exists an $A \in \mathscr{U}$ such that $A = \{a_n : n \in \omega\}$ and $a_n \in A_n$ for all n;*

(iv) *for any family $\{A_n : n \in \omega\}$, where $A_n \in \mathscr{U}$, the diagonal intersection $\Delta_{n\in\omega} A_n = \{k \in \omega : k \in \bigcap_{m<k} A_m\}$ belongs to $\mathscr{U}$;*

(v) *for any $A_n \in \mathscr{U}$, $n \in \omega$, there exists a strictly increasing function $f : \omega \to \omega$ such that $f(n+1) \in A_{f(n)}$ for each $n \in \omega$ and the range of f belongs to $\mathscr{U}$.*

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A dissection of Sirota's construction shows that $B^{\mathrm{lin}}(\omega_{\mathscr{F}})$ is extremally disconnected for any filter on ω satisfying (v). Therefore, any selective filter is an ultrafilter.

Definition

An ultrafilter $\mathscr{U}$ on ω is

- a *P*-point if, for any partition $\{A_n : n \in \omega\}$ of ω such that $A_n \notin \mathscr{U}$ for any n, there exists an $A \in \mathscr{U}$ such that $|A \cap A_n| < \aleph_0$ for any n;
- Ramsey, or selective, if, for any partition $\{A_n : n \in \omega\}$ of ω such that $A_n \notin \mathscr{U}$ for any n, there exists an $A \in \mathscr{U}$ such that $|A \cap A_n| \leqslant 1$ for any n;

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- rapid if, for any partition $\{A_n : n \in \omega\}$ of ω such that A_n is finite for any n, there exists an $A \in \mathscr{U}$ such that $|A \cap A_n| \leqslant n$ for any n $\iff$ every function $\omega \to \omega$ is majorized by the increasing enumeration of some element of $\mathscr{U}$.

CH $\implies$ $\exists$ selective ultrafilters, $P \neq Q \neq \text{selective} \neq P$

ZFC $\implies$ $\exists$ an ultrafilter which is neither a P-point nor a Q-point

Shelah: There is a model in which $\nexists$ P-point ultrafilters

Miller: In Laver's model $\nexists$ Q-points (but $\exists$ P-points)

Old problem

Does there exist a model in which there are no P-points and no Q-points?

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Denis Saveliev

Let κ be a regular cardinal.

Proposition

If $f : \kappa \to \kappa$ is such that $f^{-1}(\alpha)$ is stationary for no $\alpha \in \kappa$, then there exists a club C in κ for which $f \restriction C$ is 1-to-1.

Proof.

If $\{\alpha : f(\alpha) < \alpha\}$ is stationary, then Fodor's lemma $\implies$ there exists a stationary S such that $f \restriction S = \text{const}$, which contradicts the assumption. Hence $\{\alpha : f(\alpha) < \alpha\}$ is not stationary.
Let $A = \{\alpha : f(\alpha) \geq \alpha\}$. A contains a club. Let $B = \{\alpha : f(\beta) < \alpha \ \forall \beta < \alpha\}$. B is a club. Let $S = A \cap B$. Then S contains a club and $f \restriction S$ is 1-to-1. Indeed, if $\alpha, \beta \in S$ and $\alpha < \beta$, then $\beta \in B \implies f(\alpha) < \beta$ and $\beta \in A \implies f(\beta) \geq \beta$. □

Corollary

If $\mathscr{U}$ is an ultrafilter on κ containing the club filter, then $\mathscr{U}$ is a Q-point in the sense that, for any partition $\kappa = \sqcup A_\alpha$, $|A_\alpha| < \kappa$, there exists a $U \in \mathscr{U}$ such that $|U \cap A_\alpha| = 1$ for each α.

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Corollary

The club filter on any regular cardinal is selective.

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If there are no measurable cardinals, G is a (Boolean) extremally disconnected group, and $\{0\} = \bigcap H_n$, where each H_n is an open subgroup of G, then G contains an open (extremally disconnected) subgroup of cardinality at most the continuum.

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In Miller's model such a group contains an extremally disconnected subgroup of cardinality $\aleph_1$.

THANK YOU