Rosenthal compact spaces

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(joint work with S. Todorcevic)

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$f : X \to \mathbb{R}$ is Baire$_1$ if it is the pointwise limit of continuous functions.
$f : X \rightarrow \mathbb{R}$ is Baire$_1$ if it is the pointwise limit of continuous functions.

**Definition**

$K$ is **Rosenthal compact** if $K \subset Baire_1(X, \mathbb{R})$. 
$f : X \longrightarrow \mathbb{R}$ is Baire$_1$ if it is the pointwise limit of continuous functions.

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- $X$ is Polish.
Rosenthal compact spaces

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- \( X \) is Polish.
- We take the pointwise topology.
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Arbitrary sets \( \leftrightarrow \) Analytic sets

Arbitrary compacta \( \leftrightarrow \) Separable Rosenthal compacta
Rosenthal compact spaces are Fréchet-Urysohn (Bourgain-Fremlin-Talagrand)
Rosenthal compact spaces are Fréchet-Urysohn (Bourgain-Fremlin-Talagrand) in a Borel way (Debs)
Separable Rosenthal compact spaces

“Perfect set theorems”:

- $K$ is not metrizable $\iff K \supset C_1$ or $K \supset C_2$ or $K \supset C_3$.
- $K$ is not hereditarily separable $\iff K \supset C_2$ or $K \supset C_3$.
- $K$ is not scattered $\iff K \supset C_0$ or $K \supset C_1$ or $K \supset C_2$.
- $K$ is not a continuous image of a $4$-to-$1$ preimage of a metric space $\iff K \supset C_4$ or $\cdots$ or $K \supset C_4$.

Problems:
- $K$ is not fragmentable $\iff K \supset C_1$, $K \supset C_2^1$, $\cdots$?

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- $K$ is not fragmentable $\iff K \supseteq C_1, K \supseteq C_2, \ldots$ ?

$d$ fragments $K$ if for every $L \subset K$ there is a point of continuity $L \rightarrow (L, d)$.
“Perfect set theorems”:

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Problems:

- $K$ is not fragmentable $\iff$ $K \supset C_1$, $K \supset C_1^2$, $\cdots$ ?
- $K$ is something $\iff$ $K \supset C_1$?

$d$ fragments $K$ if for every $L \subset K$ there is a point of continuity $L \rightarrow (L, d)$. 
Localized perfect set theorem:

- If $x$ is not $G_\delta$, then $x \in C_1 \subset K$. 
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  - If $x$ is not “double $G_\delta$”, then $x \in C_1^2 \subset K$ ??
Localized perfect set theorem:

- If \( x \) is not \( G_\delta \), then \( x \in C_1 \subset K \).
- Problem: A multidimensional version?
  - If \( x \) is not “double \( G_\delta \)”, then \( x \in C_1^2 \subset K \) ??

“Double \( G_\delta \)-point”

\[ \exists \mathcal{U} \text{ countable family of open sets} \]
\[ \forall y \neq z \neq x \exists W_x, W_y, W_z \in \mathcal{U} \quad W_x \cap W_y \cap W_z = \emptyset. \]
The LUR problem

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If $K$ is separable Rosenthal compact, does $C(K)$ have a LUR renorming?

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The LUR problem

LUR problem
If $K$ is separable Rosenthal compact, does $C(K)$ have a LUR renorming?

A more set-theoretic version of this problem
If $\mathcal{B}$ is a Borel subalgebra of $\mathcal{P}(\omega)$ that does not contain $\mathcal{P}(\omega)$, is $\mathcal{B}$ $\sigma$-scattered in the pointwise topology?
Functions with countably many discontinuities

Partial answer (Haydon, Moltó, Orihuela)
Yes, if $K$ is made of functions with countably many discontinuities.
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$$CD \subset R$$

Separable compact spaces:

$R$ made of Baire$_1$ functions on Polish space

$CD$ made of functions with $\aleph_0$ discontinuities
Partial answer (Haydon, Moltó, Orihuela)

Yes, if $K$ is made of functions with countably many discontinuities.

\[ CD \subset RK \subset \mathbb{R} \]

Separable compact spaces:

- $\mathbb{R}$ made of Baire$_1$ functions on Polish space
- $RK$ made of Baire$_1$ functions on compact metric
- $CD$ made of functions with $\aleph_0$ discontinuities
Functions with countably many discontinuities

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Yes, if $K$ is made of functions with countably many discontinuities.

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Separable compact spaces:

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- $RK$ made of Baire$_1$ functions on compact metric
- $CD$ made of functions with $\aleph_0$ discontinuities

Pol, Marciszewski-Pol: $RK \neq R$ A.-Todorcevic: $CD \neq RK$
Proposition

If $K$ is $CD$, then $K$ is a Corson-to-one preimage of a metric space.
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Proof

$K \subset \{ f : X \rightarrow \mathbb{R} \}$
Proposition

If $K$ is CD, then $K$ is a Corson-to-one preimage of a metric space.

Proof

$K \subset \{ f : X \rightarrow \mathbb{R} \}$

$D \subset X$ countable dense, and $r : K \rightarrow \mathbb{R}^D$ the restriction.

Example: Take $K$ the space of all functions $f : [0,1]^2 \rightarrow \{0,1\}$ lexicographically non-decreasing.
A problem

Problem

Is every separable Rosenthal compactum a continuous image of a CD space?
Problem

Is every separable Rosenthal compactum a continuous image of a $CD$ space?

We know that this is the case for (separable supplementations of) lexicographically increasing functions $[0,1]^n \to \mathbb{R}$. But what about larger ordinals?