Universality and weak amalgamations

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Winter School in Abstract Analysis
section Set Theory & Topology
Hejnice, Jan 25 – Feb 1, 2020

Joint work with Adam Krawczyk

A. Krawczyk, W. Kubiś, *Games on finitely generated structures*, arXiv:1701.05756

Notation

Definition

Let $\mathscr{F}$ be a fixed class of finitely generated models of a fixed countable language $\mathscr{L}$; we assume $\mathscr{F}$ is closed under isomorphisms. Define

$$\sigma\mathscr{F} = \Big\{ \bigcup_{n\in\omega} X_n \colon \{X_n\}_{n\in\omega} \text{ is a chain in } \mathscr{F} \Big\}.$$

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$$\operatorname{cov}_\sigma(\mathscr{F}) = \operatorname{cf}(\sigma\mathscr{F}, \hookrightarrow).$$

Known facts

Claim

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Theorem (Fraïssé)

Assume $\mathscr{F}$ is hereditary and has both the joint embedding property and the amalgamation property. Then

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Example

Let P be a fixed nonempty set of prime numbers and let $\mathscr{F}$ be the class of all finite fields of characteristic $p \in P$. Then

$$\mathrm{cov}_\sigma(\mathscr{F}) = |P|.$$

Graphs

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A graph is k-regular if the degree of every vertex is equal to k.

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A graph is *k-regular* if the degree of every vertex is equal to k.

Claim

Every finite graph in $\mathscr{G}_k$ embeds into a finite k-regular graph.

Theorem

$\operatorname{cov}_\sigma(\mathscr{G}_2) = \aleph_0$ *and* $\operatorname{cov}_\sigma(\mathscr{G}_k) = 2^{\aleph_0}$ *for every* $k > 2$.

Amalgamations

Definition

We say that $\mathscr{F}$ has amalgamations at Z if for every two embeddings $f\colon Z \to X$, $g\colon Z \to Y$ with $X, Y \in \mathscr{F}$ there exist $W \in \mathscr{F}$ and embeddings $f'\colon X \to W$, $g'\colon Y \to W$ satisfying

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We say that $\mathscr{F}$ has the amalgamation property (AP) if it has amalgamations at every $Z \in \mathscr{F}$.

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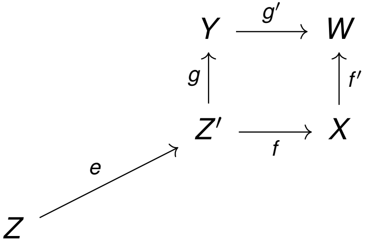
Definition (Ivanov, 1999)

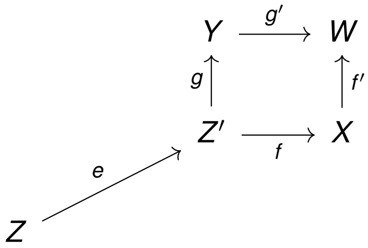
We say that $\mathscr{F}$ has the weak amalgamation property (WAP) if for every $Z \in \mathscr{F}$ there is an embedding $e\colon Z \to Z'$ with $Z' \in \mathscr{F}$, such that for every embeddings $f\colon Z' \to X$, $g\colon Z' \to Y$ there exist embeddings $f'\colon X \to W$, $g'\colon Y \to W$ satisfying

$$f' \circ f \circ e = g' \circ g \circ e.$$

CAP and WAP



CAP and WAP



Proposition

Finite graphs of vertex degree $\leqslant k$ have the CAP.

The first example of WAP and not CAP

Example (Pouzet, 1972)

Fix a linearly ordered set $(X, <)$ and let R be the following ternary relation:

$$R(x, y, z) \iff x < y,\ x < z,\ y \neq z.$$

Let $\mathscr{F}$ be the class of all finite linearly ordered set treated as models of the language $\{R\}$.
Then $\mathscr{F}$ has the WAP but not CAP.

Reference:

Reference:

- J.-F. PABION, *Relations préhomogènes*, C. R. Acad. Sci. Paris Sér. A-B 274 (1972) A529–A531.

A quote from Pabion's paper:

3° M. Pouzèt m'a communiqué l'exemple suivant de relation uniformément préhomogène et non pseudo-homogène. Sur Q, définir R (x, y, z) par $x < y$, $x < z$ et $y \neq z$.

(*) Séance du 7 février 1972.

(1) J. P. CALAIS, *Comptes rendus*, 265, série A, 1967, p. 2.

(2) R. FRAÏSSÉ, *Cours de Logiques mathématiques*, I, Gauthiers-Villars, Paris, 1967, deuxième édition 1971.

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(4) P. LINSDTROM, *Theoria*, 30, 1964, p. 183-196.

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Theorem

Let $\mathscr{F}$ be as above and assume that $\mathrm{cov}_\sigma(\mathscr{F}) < 2^{\aleph_0}$. *Then $\mathscr{F}$ has the weak amalgamation property.*

Proof.

Suppose $\mathscr{F}$ fails the WAP.

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We build a Cantor tree $\{A_s\}_{s \in 2^{<\omega}} \subseteq \mathscr{F}$ such that A_s and $A_{s^\frown 0}, A_{s^\frown 1}$ witness the failure of WAP for each $s \in 2^{<\omega}$.

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Given $\sigma \in 2^\omega$, define

$$A_\sigma = \bigcup_{n \in \omega} A_{\sigma \restriction n}.$$

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Choose $\sigma \neq \tau$ such that A_σ and A_τ are contained in a fixed $M \in \sigma\mathscr{F}$.

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Let $s = \sigma \cap \tau$. Then $A_{s^\frown 0}, A_{s^\frown 1}$ can be amalgamated inside M.

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A contradiction. □

The Banach-Mazur game

Definition (BM ($\mathscr{F}, \mathfrak{M}$))

Let $\mathscr{F}$ be as above, $\mathfrak{M} \subseteq \sigma\mathscr{F}$. Two players, Eve and Adam, alternately choose bigger and bigger models from $\mathscr{F}$, building a chain

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$$

Of course, Eve starts the game.

The Banach-Mazur game

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Of course, Eve starts the game.
Adam wins if $\bigcup_{n\in\omega} A_n$ embeds into some $M \in \mathfrak{M}$. Otherwise Eve wins.

Theorem

Let $\mathscr{F}$ be as above and assume Adam has a winning strategy in $\mathrm{BM}(\mathscr{F}, \mathfrak{M})$, *where* $|\mathfrak{M}| < 2^{\aleph_0}$. *Then $\mathscr{F}$ has the weak amalgamation property.*

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Corollary

Assume $\mathscr{F}$ has the joint embedding property and countably many isomorphic types. The following conditions are equivalent:

(a) *There is $\mathfrak{M} \subseteq \sigma\mathscr{F}$ with $|\mathfrak{M}| < 2^{\aleph_0}$ such that Adam has a winning strategy in* $\mathrm{BM}(\mathscr{F}, \mathfrak{M})$.

(b) *$\mathscr{F}$ has the weak amalgamation property.*

(c) *There is $U \in \sigma\mathscr{F}$ such that Adam has a winning strategy in* $\mathrm{BM}(\mathscr{F}, \{U\})$.

Theorem

Assume $\mathscr{F}$ fails the weak amalgamation property. Then Eve has a winning strategy in $\mathrm{BM}(\mathscr{F}, \{V\})$ *for every* $V \in \sigma\mathscr{F}$.

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Problem

Find a class $\mathscr{F}$ of finite models of a fixed finite language such that consistently

$$\aleph_0 < \mathrm{cov}_\sigma(\mathscr{F}) < 2^{\aleph_0}.$$

Further examples

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Fix a nontrivial subgroup S of $(\mathbb{R}, +)$. Let $\mathfrak{M}_S$ be the class of all finite metric spaces with distances in S.

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Theorem

If S is countable then $\mathrm{cov}_\sigma(\mathfrak{M}_S) = 1$, *otherwise*

$$\mathrm{cov}_\sigma(\mathfrak{M}_S) = \mathrm{cf}\left([\kappa]^{\aleph_0}, \subseteq\right),$$

where $\kappa = |S|$.

Example

Let $\mathscr{F}$ be the class of all finite graphs in which different cycles of equal length do not have a common edge.
Then $\mathscr{F}$ fails the weak amalgamation property.

A. Krawczyk, W. Kubiś, *Games on finitely generated structures*, arXiv:1701.05756

A. Krawczyk, A. Kruckman, W. Kubiś, A. Panagiotopoulos, *Examples of weak amalgamation classes*, arXiv:1907.09577

Thank you for your attention!