Universality and weak amalgamations

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Let $\mathcal{F}$ be a fixed class of finitely generated models of a fixed countable language $\mathcal{L}$; we assume $\mathcal{F}$ is closed under isomorphisms. Define

$$\sigma \mathcal{F} = \left\{ \bigcup_{n \in \omega} X_n : \{X_n\}_{n \in \omega} \text{ is a chain in } \mathcal{F} \right\}.$$
Notation

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$$\text{cov}_{\sigma}(\mathcal{F}) = \text{cf}(\sigma(\mathcal{F}), \hookrightarrow).$$
Known facts

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Theorem (Fraïssé)
Assume \( \mathcal{F} \) is hereditary and has both the joint embedding property and the amalgamation property. Then

\[ \text{cov}_\sigma(\mathcal{F}) = 1. \]
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Example
Let \( P \) be a fixed nonempty set of prime numbers and let \( \mathcal{F} \) be the class of all finite fields of characteristic \( p \in P \). Then
\[ \text{cov}_\sigma(\mathcal{F}) = |P|. \]
Example

Fix $k > 1$ and let $G_k$ be the class of all finite graphs of vertex degree $\leq k$. 
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A graph is $k$-regular if the degree of every vertex is equal to $k$.

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Graphs

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Theorem
$\text{cov}_\sigma(\mathcal{G}_2) = \aleph_0$ and $\text{cov}_\sigma(\mathcal{G}_k) = 2^{\aleph_0}$ for every $k > 2$. 
Amalgamations

Definition

We say that \( F \) has amalgamations at \( Z \) if for every two embeddings \( f: Z \to X, g: Z \to Y \) with \( X, Y \in F \) there exist \( W \in F \) and embeddings \( f': X \to W, g': Y \to W \) satisfying

\[ f' \circ f = g' \circ g. \]
Amalgamations

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Definition

We say that $\mathcal{F}$ has the amalgamation property (AP) if it has amalgamations at every $Z \in \mathcal{F}$. 
Weakenings of amalgamation

Definition
We say that $F$ has the cofinal amalgamation property (CAP) if for every $Z \in F$ there is an embedding $e: Z \rightarrow Z'$ such that $F$ has amalgamations at $Z'$.

Definition (Ivanov, 1999)
We say that $F$ has the weak amalgamation property (WAP) if for every $Z \in F$ there is an embedding $e: Z \rightarrow Z'$ with $Z' \in F$, such that for every embeddings $f: Z' \rightarrow X$, $g: Z' \rightarrow Y$ there exist embeddings $f': X \rightarrow W$, $g': Y \rightarrow W$ satisfying $f' \circ f \circ e = g' \circ g \circ e$. 

W.Kubiš (http://www.math.cas.cz/kubis/) Universality vs. WAP 31 January 2020 7/18
Weakenings of amalgamation

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Proposition
Finite graphs of vertex degree \( \leq k \) have the CAP.

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Proposition

Finite graphs of vertex degree $\leq k$ have the CAP.
The first example of WAP and not CAP

Example (Pouzet, 1972)

Fix a linearly ordered set \((X, <)\) and let \(R\) be the following ternary relation:

\[ R(x, y, z) \iff x < y, \ x < z, \ y \neq z. \]

Let \(\mathcal{F}\) be the class of all finite linearly ordered set treated as models of the language \(\{R\}\).

Then \(\mathcal{F}\) has the WAP but not CAP.
Reference:


A quote from Pabion’s paper:

3° M. Pouzet m’a communiqué l’exemple suivant de relation uniformément préhomogène et non pseudo-homogène. Sur $Q$, définir $R(x, y, z)$ par $x < y, x < z$ et $y = z$.

(*) Séance du 7 février 1972.

Université Claude Bernard,
Mathématiques,
43, boulevard du Onze-Novembre 1918,
69-Villeurbanne, Rhône.
Theorem

Let $\mathcal{F}$ be as above and assume that $\text{cov}_\sigma(\mathcal{F}) < 2^{\aleph_0}$. Then $\mathcal{F}$ has the weak amalgamation property.
Proof.

Suppose $\mathcal{F}$ fails the WAP.
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Suppose $\mathcal{F}$ fails the WAP. We build a Cantor tree $\{A_s\}_{s \in 2^{<\omega}} \subseteq \mathcal{F}$ such that $A_s$ and $A_s \upharpoonright 0$, $A_s \upharpoonright 1$ witness the failure of WAP for each $s \in 2^{<\omega}$. Given $\sigma \in 2^{\omega}$, define $A_{\sigma} = \bigcup_{n \in \omega} A_{\sigma} \upharpoonright n$. Choose $\sigma \neq \tau$ such that $A_{\sigma}$ and $A_{\tau}$ are contained in a fixed $M \in \mathcal{F}$. Let $s = \sigma \cap \tau$. Then $A_s \upharpoonright 0$, $A_s \upharpoonright 1$ can be amalgamated inside $M$. A contradiction.
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A contradiction. 

$\square$
The Banach-Mazur game

**Definition (BM (\( \mathcal{F}, \mathcal{M} \)) )**

Let \( \mathcal{F} \) be as above, \( \mathcal{M} \subseteq \sigma \mathcal{F} \). Two players, Eve and Adam, alternately choose bigger and bigger models from \( \mathcal{F} \), building a chain

\[
A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots
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Of course, Eve starts the game.
The Banach-Mazur game

**Definition (BM ($\mathcal{F}$, $\mathcal{M}$))**

Let $\mathcal{F}$ be as above, $\mathcal{M} \subseteq \sigma \mathcal{F}$. Two players, Eve and Adam, alternately choose bigger and bigger models from $\mathcal{F}$, building a chain

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$$

Of course, Eve starts the game. **Adam wins** if $\bigcup_{n \in \omega} A_n$ embeds into some $M \in \mathcal{M}$. Otherwise Eve wins.
Theorem

Let $\mathcal{F}$ be as above and assume Adam has a winning strategy in $\text{BM} (\mathcal{F}, M)$, where $|M| < 2^{\aleph_0}$. Then $\mathcal{F}$ has the weak amalgamation property.
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Corollary
Assume $\mathcal{F}$ has the joint embedding property and countably many isomorphic types. The following conditions are equivalent:

(a) There is $M \subseteq \sigma \mathcal{F}$ with $|M| < 2^{\aleph_0}$ such that Adam has a winning strategy in $\text{BM} (\mathcal{F}, M)$.

(b) $\mathcal{F}$ has the weak amalgamation property.

(c) There is $U \in \sigma \mathcal{F}$ such that Adam has a winning strategy in $\text{BM} (\mathcal{F}, \{U\})$. 
Theorem

Assume $\mathcal{F}$ fails the weak amalgamation property. Then Eve has a winning strategy in $BM(\mathcal{F}, \{V\})$ for every $V \in \sigma \mathcal{F}$. 

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Problem

Find a class $\mathcal{F}$ of finite models of a fixed finite language such that consistently

$$\aleph_0 < \text{cov}_\sigma(\mathcal{F}) < 2^{\aleph_0}.$$
Further examples

Example

Fix a nontrivial subgroup $S$ of $(\mathbb{R}, +)$. Let $M_S$ be the class of all finite metric spaces with distances in $S$.

Theorem

If $S$ is countable then $\text{cov}_\sigma(M_S) = 1$, otherwise $\text{cov}_\sigma(M_S) = \text{cf}(\mathbb{[}\kappa\mathbb{]}, \subseteq)$, where $\kappa = |S|$.
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W. Kubiš (http://www.math.cas.cz/kubis/)
Further examples

Example

Fix a nontrivial subgroup $S$ of $\mathbb{R}, +$. Let $\mathcal{MS}$ be the class of all finite metric spaces with distances in $S$.

Theorem

If $S$ is countable then $\text{cov}_\sigma(\mathcal{MS}) = 1$, otherwise

$$\text{cov}_\sigma(\mathcal{MS}) = \text{cf} \left( [\kappa]^{\omega_0}, \subseteq \right),$$

where $\kappa = |S|$. 
Example

Let $\mathcal{F}$ be the class of all finite graphs in which different cycles of equal length do not have a common edge. Then $\mathcal{F}$ fails the weak amalgamation property.


Thank you for your attention!