Unbounded towers and products

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(joint work with P. Szewczak)

Covering properties

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Local properties of $C_p(X)$

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A space is α_4, if for each x and each sequence $\{\, A_n : n \in \mathbb{N} \,\}$ of nontrivial sequences converging to x, there is a sequence B, converging to x, such that $B \cap A_n \neq \emptyset$ for infinitey many n

The space of continuous functions

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$$\begin{array}{ccc}
\underset{\mathfrak{b}}{\mathsf{S}_1(\Gamma, \Gamma)} & & \\
\uparrow & & \\
\underset{\mathfrak{p}}{\mathsf{S}_1(\Omega, \Gamma)} & \longrightarrow & \underset{\mathrm{cov}(\mathcal{M})}{\mathsf{S}_1(\Omega, \Omega)}
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κ-unbounded tower set $= X \cup \mathrm{Fin}$

Theorem 1 (Tsaban)

A $\mathfrak{p}$-unbounded tower set is $\mathsf{S}_1(\Omega, \Gamma)$

Products of $S_1(\Omega, \Gamma)$ sets

Theorem 2 (Miller, Tsaban, Zdomskyy)

Assuming CH, there are $S_1(\Omega, \Gamma)$ *sets* X *and* Y *such that* $X \times Y$ *is not Menger*

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Theorem 3 (Miller, Tsaban, Zdomskyy)

The product of an ω_1-unbounded tower set with an $S_1(\Omega, \Gamma)$ set is $S_1(\Omega, \Gamma)$.

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Theorem 4 (Szewczak, MW)

The product of a $\mathfrak{p}$-unbounded tower set with an $\mathsf{S}_1(\Omega,\Gamma)$ set is $\mathsf{S}_1(\Omega,\Omega)$ and $\mathsf{S}_1(\Gamma,\Gamma)$ in all finite powers.

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Theorem 5 (Miller, Tsaban)

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Theorem 5 (Miller, Tsaban)

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Theorem 6 (Miller, Tsaban, Zdomskyy)

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Theorem 7 (Szewczak, MW)

A $\mathfrak{b}$-unbounded tower set is $\mathsf{S}_1(\Gamma, \Gamma)$ in all finite powers.
A product of a finite power of a $\mathfrak{b}$-unbounded tower set with a Sierpiński set is $\mathsf{S}_1(\Gamma, \Gamma)$.